133. *Bounded Variation Property of a Measure*

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1. Introduction. For an integral structure $\Gamma=(\Lambda;\mathcal{S},\mathcal{G},Q)$ defined in [3], we shall discuss in this paper a certain type of bounded variation property of a pre-measure $\mu\in Q$. Through the discussion, some properties of the 'indefinite integral' $\sigma(\cdot,f,\mu)$, where σ is an integral with respect to Γ, and a theorem similar to Lebesgue's bounded convergence theorem will be obtained.

2. Bounded variation property.

Assumption 1. *M is a set and $\mathcal{S}$ is a ring of subsets of M. G is a topological additive group and μ is a G-valued pre-measure on $\mathcal{S}$.*

Let us denote by $\mathcal{V}$ the system of neighbourhoods of $0\in G$.

The pre-measure μ is *locally s-bounded* if, for any $X\in\mathcal{S}$ and $X_i\in\mathcal{S}$, $i=1,2,\cdots$, such that $X_jX_k=0$ $(j\neq k)$, and for any $V\in\mathcal{V}$, there exists a positive integer n such that $\mu(XX_i)\in V$ for any $i\geqq n$.

Proposition 1. *If $\mathcal{S}$ is a pseudo-σ-ring and μ is a measure, then μ is locally s-bounded.*

Proof. Let X and X_i, $i=1,2,\cdots$, be elements of $\mathcal{S}$ such that $X_jX_k=0$ $(j\neq k)$ and V an element of $\mathcal{V}$. Since $\mathcal{S}$ is a pseudo-σ-ring, $Y_n=\bigcup_{i=n}^{\infty}XX_i$ is an element of $\mathcal{S}$ for each $n=1,2,\cdots$. Since μ is a measure, it follows from $Y_n\downarrow 0$ $(n\to\infty)$ that $\mu(Y_n)\to 0$ $(n\to\infty)$. Hence, for an element V_0 of $\mathcal{V}$ such that $V_0-V_0\subset V$, we have a positive integer n such that $\mu(Y_i)\in V_0$ for any $i\geqq n$. For this n and for any $i\geqq n$, we have $\mu(XX_i)=\mu(Y_i-Y_{i+1})=\mu(Y_i)-\mu(Y_{i+1})\in V_0-V_0\subset V$, which proves the proposition.

For an element V of $\mathcal{V}$, an element X of $\mathcal{S}$ is *of V-variation* if $\mu(XY)\in V$ for any $Y\in\mathcal{S}$.

Then the following is easily seen:

Proposition 2. *If an element X of $\mathcal{S}$ is of V-variation with $V\in\mathcal{V}$, then XY is of V-variation for any $Y\in\mathcal{S}$.*

Proposition 3. *Suppose that μ is a locally s-bounded measure and $X_i\downarrow 0$ $(i\to\infty)$ for $X_i\in\mathcal{S}$, $i=1,2,\cdots$. Then for any $V\in\mathcal{V}$ there exists a positive integer n such that X_n is of V-variation.*

Proof. Let us assume that no X_i is of V-variation. Let V_0 be an element of $\mathcal{V}$ such that $2V_0\subset V$. Put $i_0=1$ and assume that a positive integer i_{n-1} is defined. Then we have an element $Y_{i_{n-1}}$ of $\mathcal{S}$ such that $Y_{i_{n-1}}\subset X_{i_{n-1}}$ and $\mu(Y_{i_{n-1}})\notin V$. Since $Y_{i_{n-1}}X_j\downarrow 0$ $(j\to\infty)$ implies $\mu(Y_{i_{n-1}}X_j)$

$\to 0$ $(j\to\infty)$, there exists an $i_n > i_{n-1}$ such that $\mu(Y_{i_{n-1}}X_{i_n}) \in V_0$. Putting $Z_n = Y_{i_{n-1}} + Y_{i_{n-1}}X_{i_n}$, we have positive integers i_n and $Z_n \in \mathcal{S}$, $n=1,2,\cdots$, defined inductively. It follows from $V \not\ni \mu(Y_{i_{n-1}}) = \mu(Z_n) + \mu(Y_{i_{n-1}}X_{i_n}) \in \mu(Z_n) + V_0$ that $\mu(Z_n) \notin V_0$. The relations $Z_n X_{i_n} = 0$ and $Z_m \subset Y_{i_{m-1}} \subset X_{i_{m-1}} \subset X_{i_n}$, where $m > n$, imply that $Z_j Z_k = 0$ $(j \neq k)$. Thus the locally s-boundedness of μ implies the existence of n such that $\mu(Z_n) = \mu(X_1 Z_n) \in V_0$. This is a contradiction and hence our proposition is proved.

Assumption 2. *σ is an integral with respect to an integral structure $(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ with $\Lambda = (M, G, K, J)$ and μ is an element of Q. Further*

1) *$\mathcal{G}$ is a subgroup of the fundamental functional group of Λ determined by $\mathcal{S}$.*

2) *For each $k \in K$, the map φ_k of G into J defined by $\varphi_k(g) = g\cdot k$ is continuous.*

Let us denote by $\mathcal{W}$ the system of neighbourhoods of $0 \in J$.

Lemma 1. *Suppose that $X \in \mathcal{S}$ and that B is a totally bounded subset of K. Then for any $W \in \mathcal{W}$ there exists an element V of $\mathcal{V}$ satisfying the condition: if Y is an element of V-variation in $\mathcal{S}$ and if $Y \subset X$, then it follows that $\sigma(Y, f, \mu) \in W$ for any $f \in \mathcal{G}$ such that $f(Y) \subset B$.*

Proof. (P1) Let $\mathcal{F}$ be the total functional group of Λ and $\mathcal{G}_0$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}K$. Denote by $\mathcal{J}$ the abstract integral derived from σ relative to μ. Then, for a fixed $W_0 \in \mathcal{W}$ such that $2W_0 \subset W$, there exists a neighbourhood U of $0 \in K$ such that $\mathcal{J}(X, \tilde{U} \cap \mathcal{G}) \subset W_0$. Here we write $\tilde{U'} = \{f \mid f \in \mathcal{F}, f(M) \subset U'\}$ for each $U' \subset K$.

(L1) Let U_0 be a neighbourhood of $0 \in K$ such that $-U_0 = U_0$ and $3U_0 \subset U$. Since B is totally bounded, there exist $b_j \in K$, $j = 1, 2, \cdots, n$, such that $B \subset \bigcup_{j=1}^n (b_j + U_0)$.

(P2) For a fixed $W_1 \in \mathcal{W}$ such that $nW_1 \subset W_0$, the continuity of the map $G \ni g \to g\cdot b_j \in J$ implies the existence of $V_j \in \mathcal{V}$ such that $V_j \cdot b_j \subset W_1$ for each j.

(L2) Put $V = \bigcap_{j=1}^n V_j \in \mathcal{V}$ and let Y be an element of V-variation in $\mathcal{S}$ such that $Y \subset X$. Then it suffices to show that $\mathcal{J}(Y, f) \in W$ for any $f \in \mathcal{G}$ such that $f(Y) \subset B$. Putting $g = Yf$ we have $g \in \overline{\mathcal{G}}_0 \cap \mathcal{G}$ and this implies the existence of $\psi \in \mathcal{G}_0$ such that $g - \psi \in \tilde{U}_0$. We can write $\psi = \sum_{k=1}^m Z_k a_k$ for some $Z_k \in \mathcal{S}$ and $a_k \in K$, $k = 1, 2, \cdots, m$, such that $Z_k Z_{k'} = 0$ $(k \neq k')$. It may be assumed that $\sum_{k=1}^m Z_k = Y$ and $Z_k \neq 0$ for each k. Let z_k be an element of Z_k. Then we have $g(z_k) = a_k + \{g(z_k) - \psi(z_k)\} \in a_k + U_0$ and the relation $g(z_k) = f(z_k) \in B$ implies the existence of j_k with $1 \leqq j_k \leqq n$ such that $g(z_k) \in b_{j_k} + U_0$. Thus it follows that $a_k - b_{j_k} \in 2U_0$. Putting $\varphi = \sum_{k=1}^m Z_k b_{j_k}$ we have $\varphi \in \mathcal{G}_0 \subset \mathcal{G}$ and $\psi - \varphi = \sum_{k=1}^m Z_k(a_k - b_{j_k}) \in \widetilde{2U_0}$, which implies $g - \varphi = (g - \psi) + (\psi - \varphi) \in \widetilde{3U_0} \subset \tilde{U}$.

Put $P_j=\sum_{j_k=j} Z_k$, $j=1,2,\cdots,n$. Then it follows that $\varphi=\sum_{j=1}^n P_j b_j$ and the V-variation property of Y implies that $\mu(YP_j)\in V\subset V_j$ for each $j=1,2,\cdots,n$.

(P3) Since $g,\varphi\in\mathcal{G}$ and $g-\varphi\in\tilde{U}$ imply $Y(g-\varphi)\in\tilde{U}\cap\mathcal{G}$, it follows that $\mathcal{I}(Y,g-\varphi)=\mathcal{I}(XY,g-\varphi)=\mathcal{I}(X,Y(g-\varphi))\in W_0$. Further we have $\mathcal{I}(Y,\varphi)=\mathcal{I}(Y,\sum_{j=1}^n P_j b_j)=\sum_{j=1}^n \mathcal{I}(YP_j,b_j)=\sum_{j=1}^n \mu(YP_j)\cdot b_j\in\sum_{j=1}^n V_j\cdot b_j \subset nW_1\subset W_0$. Hence we have $\mathcal{I}(Y,f)=\mathcal{I}(Y,g)=\mathcal{I}(Y,g-\varphi)+\mathcal{I}(Y,\varphi)\in 2W_0\subset W$.

Thus Lemma 1 is proved. Since $f(X)$ is totally bounded for $f\in\mathcal{G}$ and $X\in\mathcal{S}$ (Theorem 3.2 in [4]), then follows Corollary 1 below, which implies the absolute continuity, in a sense, of the indefinite integral $\sigma(\cdot,f,\mu)$.

Corollary 1. *Let f be an element of $\mathcal{G}$. Then for any $X\in\mathcal{S}$ and $W\in\mathcal{W}$ there exists an element V of $\mathcal{V}$ satisfying the condition: if an element $Y\in\mathcal{S}$ contained in X is of V-variation then it follows that $\sigma(Y,f,\mu)\in W$.*

Corollary 2. *Suppose that μ is a locally s-bounded measure and $X_i\downarrow 0$ $(i\to\infty)$ for $X_i\in\mathcal{S}$, $i=1,2,\cdots$. Then for any totally bounded subset B of K and for any $W\in\mathcal{W}$ there exists a positive integer n satisfying the condition: for any $Y\in\mathcal{S}$ such that $Y\subset X_n$ and for any $f\in\mathcal{G}$ such that $f(Y)\subset B$ it holds that $\sigma(Y,f,\mu)\in W$.*

Proof. For the sets $X=X_1\in\mathcal{S}$, $B\subset K$ and $W\in\mathcal{W}$, let V be an element of $\mathcal{V}$ satisfying the condition stated in Lemma 1. Then Proposition 3 implies the existence of n such that X_n is of V-variation. The relations $Y\in\mathcal{S}$ and $Y\subset X_n\subset X$ imply that Y is of V-variation and thus the relations $f\in\mathcal{G}$ and $f(Y)\subset B$ imply $\sigma(Y,f,\mu)\in W$.

Let us show that the indefinite integral $\sigma(\cdot,f,\mu)$ is a measure if so is μ:

Proposition 4. *Suppose that μ is a measure and $X_i\downarrow 0$ $(i\to\infty)$ for $X_i\in\mathcal{S}$, $i=1,2,\cdots$. Then for any $f\in\mathcal{G}$ it holds that $\sigma(X_i,f,\mu)\to 0$ $(i\to\infty)$.*

Proof. For any $W\in\mathcal{W}$, it suffices to show the existence of a positive integer l such that $\sigma(X_i,f,\mu)\in W$ for each $i\geqq l$. For $X=X_1$ let us consider the neighbourhoods W_0 and U stated in (P1) in the proof of Lemma 1. Putting $g=Xf$ we have $g\in\overline{\mathcal{G}}_0\cap\mathcal{G}$, which implies the existence of $\varphi\in\mathcal{G}_0\subset\mathcal{G}$ such that $g-\varphi\in\tilde{U}$. Here we may write $\varphi=\sum_{j=1}^n P_j b_j$ with $P_j\in\mathcal{S}$ and $b_j\in K$, $j=1,2,\cdots,n$, such that $P_jP_{j'}=0$ $(j\neq j')$. Now let us consider the neighbourhoods W_1 and V_j, $j=1,2,\cdots,n$, stated in (P2). For each j, we have $X_iP_j\downarrow 0$ $(i\to\infty)$ and this implies the existence of l_j such that $\mu(X_iP_j)\in V_j$ for any $i\geqq l_j$. Put $l=\max(l_1,l_2,\cdots,l_n)$ and for any fixed $i\geqq l$ put $Y=X_i$. Then we are to show that $\mathcal{I}(Y,f)\in W$ and this follows from the arguments in (P3).

Theorem 1. *Suppose that $\mathcal{S}$ is a pseudo-σ-ring and μ is a measure. Let X be an element of $\mathcal{S}$ and let f and f_i, $i=1,2,\cdots$, be elements of $\mathcal{G}$ such that: each f_i-f is measurable*[1] *and $\bigcup_{i=1}^{\infty} f_i(X)$ is totally bounded. Then the pointwise convergence $f_i(x)\to f(x)$ $(i\to\infty)$ implies the convergence $\sigma(X, f_i, \mu)\to\sigma(X, f, \mu)$ $(i\to\infty)$.*

Proof. The subset $f(X)$ of the closure $\bar{B}$ of the totally bounded set $B=\bigcup_{i=1}^{\infty} f_i(X)$ is also totally bounded. Hence the subset $\bigcup_{i=1}^{\infty}((f_i-f)(X))$ of the set $\{u-v \mid u\in B, v\in \bar{B}\}$ is totally bounded. This implies that we may assume $f=0$.

Denote by $\mathcal{J}$ the abstract integral derived from σ relative to μ and let W be an element of $\mathcal{W}$. Then it is sufficient to show the existence of a positive integer n such that $\mathcal{J}(X, f_i)\in W$ for each $i\geqq n$.

For a fixed $W_0\in\mathcal{W}$ such that $2W_0\subset W$ there exists an open neighbourhood U of $0\in K$ such that $\mathcal{J}(X,\tilde{U})\subset W_0$, where $\tilde{U}=\{g \mid g\in\mathcal{G}, g(M)\subset U\}$. For each i, the measurability of f_i implies $f_i^{-1}(U)\cap X\in\mathcal{S}$. Hence, putting $X_i=\{x \mid x\in X, f_i(x)\notin U\}$, we have $X_i=X-f_i^{-1}(U)=X-(f_i^{-1}(U)\cap X)$ and this implies $X_i\in\mathcal{S}$. For $Y_j=\bigcup_{i=j}^{\infty} X_i$, $j=1,2,\cdots$, it holds that $X\supset Y_j\in\mathcal{S}$ and $Y_1\supset Y_2\supset\cdots$. Now we assert that $Y_j\downarrow 0$ $(j\to\infty)$. Otherwise there exists an element y of $\bigcap_{j=1}^{\infty} Y_j$. Then for each j there exists $i_j\geqq j$ such that $f_{i_j}(y)\notin U$ and this contradicts the convergence of $f_i(y)$ to 0. Since Proposition 1 implies that μ is locally s-bounded, and since $B=\bigcup_{i=1}^{\infty} f_i(X)$ is totally bounded, Corollary 2 to Lemma 1 implies the existence of a positive integer n satisfying the condition: for any $Z\in\mathcal{S}$ such that $Z\subset Y_n$ and for any $h\in\mathcal{G}$ such that $h(Z)\subset B$ it holds that $\mathcal{J}(Z, h)\in W_0$. Then we are to show that $\mathcal{J}(X, f_i)\in W$ for each $i\geqq n$.

It follows from $f_i(Y_n)\subset f_i(X)\subset B$ that $\mathcal{J}(Y_n, f_i)\in W_0$. Since $Y_n\supset Y_i\supset X_i$ implies $X-Y_n\subset X-X_i$, we have $f_i(x)\in U$ for any $x\in X-Y_n$ and this implies $(X-Y_n)f_i\in\tilde{U}$. Thus it follows that $\mathcal{J}(X-Y_n, f_i)=\mathcal{J}(X,(X-Y_n)f_i)\in W_0$ and hence we have $\mathcal{J}(X, f_i)=\mathcal{J}(X-Y_n, f_i)+\mathcal{J}(Y_n, f_i)\in 2W_0\subset W$, which proves the theorem.

References

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1) If K satisfies the first condition of countability, then the measurability of f_i-f follows from the fact that $\mathcal{S}$ is a pseudo-σ-ring (Corollary 3 to Theorem 3.3 in [4]).