131. On Normal Approximate Spectrum. VI

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1. Introduction. For a unital C^*-algebra $\mathfrak{A}$, the connectedness of the set $G[\mathfrak{A}]$ of all regular members of $\mathfrak{A}$ is discussed in several occasions: In an early stage, Kakutani observed in [14; pp. 280–281], $G[\mathfrak{A}]$ is connected if $\mathfrak{A}$ is the algebra $\mathfrak{B}(\mathfrak{H})$ of all operators acting on a Hilbert space $\mathfrak{H}$. Kuiper [13] proved that the homotopy group $\pi_m(G[\mathfrak{A}])$ vanishes for all m if $\mathfrak{A}=\mathfrak{B}(\mathfrak{H})$. Breuer [1] generalized Kuiper's theorem for every semifinite properly infinite factor. However, if $\mathfrak{A}$ is not large, then the situation changes. Kakutani pointed out in [14; p. 294], the set of all regular elements of the algebra $C(S^1)$ of all continuous functions on the unit circle S^1 is not connected: $G[C(S^1)]$ has infinitely many components each of which contains one of

$$e_n(s)=e^{2\pi ins} \qquad (n=0, \pm 1, \pm 2, \cdots). \tag{1}$$

In the present note, the connectedness of $G[\mathfrak{A}]$ for a general C^*-algebra $\mathfrak{A}$ is considered in § 2, where some theorems of Cordes and Labrousse [6] are given alternative proofs, and they are combined with a theorem of Royden [15]. In § 3, a unital C^*-algebra generated by an operator will be discussed; theorems on the algebraic theory of Fredholm operators, discussed by Breuer-Cordes [2] and Coburn-Lebow [4], are applied, and some elementary properties of the index are proved. In § 4, the unital C^*-algebra generated by the unilateral shift is discussed to illustrate these considerations. In §§ 3–4, the normal approximate spectrum of the generator plays a central role.

2. Connectedness. A member A of $G[\mathfrak{A}]$ of a unital C^*-algebra $\mathfrak{A}$ is *homotopic* (in $G[\mathfrak{A}]$) with $B \in G[\mathfrak{A}]$ if there is a continuous way $A_t (0 \leqq t \leqq 1)$ in $G[\mathfrak{A}]$ with $A_0=A$ and $A_1=B$.

The following two theorems are obtained in [6] with somewhat different proofs:

Theorem 1 (Cordes-Labrousse). *If $H \in \mathfrak{A}$ is an invertible and positive element, then H is homotopic with* 1.

Define

$$H_t=t+(1-t)H \qquad (0 \leqq t \leqq 1). \tag{2}$$

Then H_t is positive and invertible by the Gelfand representation (of the unital C^*-algebra generated by H). H_t is continuous in t with $H_0=H$ and $H_1=1$; hence H is homotopic with 1.

Theorem 2 (Cordes-Labrousse). *If $A \in G[\mathfrak{A}]$ and*

(3) $$A = UH$$
is the polar decomposition of A, then A is homotopic with U.

By the polar decomposition of operators, the invertibility of H follows from that of A; hence $U = AH^{-1} \in \mathfrak{A}$ is also invertible. By (2), A is homotopic with U via

(4) $$A_t = UH_t\ ,$$
which is continuous in t with $A_0 = A$ and $A_1 = U$.

Let $U[\mathfrak{A}]$ be the group of all unitary members of $\mathfrak{A}$. Suppose that $G_1[\mathfrak{A}]$ (resp. $U_1[\mathfrak{A}]$) is the arcwise connected principal component of $G[\mathfrak{A}]$ (resp. $U[\mathfrak{A}]$) containing 1. Then $G_1[\mathfrak{A}]$ (resp. $U_1[\mathfrak{A}]$) is a normal subgroup.

Theorem 3. $U_1[\mathfrak{A}] = U[\mathfrak{A}] \cap G_1[\mathfrak{A}]$.

If $U \in G[\mathfrak{A}]$ is homotopic in $G[\mathfrak{A}]$ with 1 via A_t, and if $A_t = U_tH_t$ is the polar decomposition of A_t for every t, then H_t is continuous in t, and so $U_t = A_tH_t^{-1}$ is continuous in t, with $U_0 = U$ and $U_1 = 1$; hence U is homotopic with 1 in $U[\mathfrak{A}]$, so that the theorem is proved.

Theorem 4. $U[\mathfrak{A}]/U_1[\mathfrak{A}]$ *is isomorphic to* $G[\mathfrak{A}]/G_1[\mathfrak{A}]$.

If U (resp. V) $\in U[\mathfrak{A}]$ is homotopic with A (resp. B) via A_t (resp. B_t), then UV is homotopic with AB via A_tB_t, which proves that the product in $G[\mathfrak{A}]/G_1[\mathfrak{A}]$ is represented by the unitary members up to homotopic.

By virtue of Theorem 4, the *cohomology* $H[\mathfrak{A}]$ of $\mathfrak{A}$ is introduced by

(5) $$H[\mathfrak{A}] = U[\mathfrak{A}]/U_1[\mathfrak{A}] = G[\mathfrak{A}]/G_1[\mathfrak{A}].$$
This name may be justified in the next section.

Let $\mathfrak{K}$ be a closed (two-sided) ideal of $\mathfrak{A}$. Then the natural homomorphism π of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{K}$ defines

(6) $$F(\mathfrak{A}\,;\mathfrak{K}) = \pi^{-1}G[\mathfrak{A}/\mathfrak{K}].$$
Each member of $F(\mathfrak{A}\,;\mathfrak{K})$ is called $\mathfrak{K}$-*Fredholm* according to a recent convention due to [4] and [5]. By [4; Theorem 2.1], the following theorem is established:

Theorem 5 (Coburn-Lebow). *If* $H[F(\mathfrak{A}\,;\mathfrak{K})]$ *is the set of all (arcwise connected) components of* $F(\mathfrak{A}\,;\mathfrak{K})$ *with the natural composition, then* $H[F(\mathfrak{A}\,;\mathfrak{K})]$ *is isomorphic to* $H[\mathfrak{A}/\mathfrak{K}]$.

In the remainder of the note, it will be assumed that $\mathfrak{A}/\mathfrak{K}$ *is abelian.* Coburn and Lebow pointed out in [4; p. 579], the following theorem follows from [15; § 7]:

Theorem 6 (Royden). $H[\mathfrak{A}/\mathfrak{K}]$ *is isomorphic to the first Čech cohomology group* $H^1(X, Z)$ *if* $\mathfrak{A}/\mathfrak{K}$ *is abelian, where* X *is the character space of all characters of* $\mathfrak{A}/\mathfrak{K}$ *(equipped with the weak* topology) and* Z *is the additive group of all integers.*

Hence $H[F(\mathfrak{A}\,;\mathfrak{K})]$ is isomorphic to $H^1(X, Z)$.

3. Index. For an operator T on a (separable) Hilbert space $\mathfrak{H}$, a complex number λ is a *normal approximate propervalue* of T if there is a sequence $\{x_n\}$ of unit vectors such that

(7) $$\|(T-\lambda)x_n\| \to 0 \quad \text{and} \quad \|(T-\lambda)^* x_n\| \to 0.$$

The *normal approximate spectrum* $\pi_n(T)$ is the set of all normal approximate propervalues, which is a (possibly void) compact set in the complex plane, cf. [7].

If $\mathfrak{A}$ is the unital C^*-algebra generated by T, then it is proved in [7] and [12] that $\lambda \in \pi_n(T)$ if and only if there is a character ϕ of $\mathfrak{A}$ such as

(8) $$\phi(T) = \lambda.$$

$\mathfrak{A}$ contains the *pseudoradical* $\mathfrak{R}$ by which $\mathfrak{A}/\mathfrak{R}$ is isomorphic to $C(\pi_n(T))$, cf. [9; § 5].

If $T \in F(\mathfrak{A}; \mathfrak{R})$, then T is called briefly a *T-Fredholm operator*. The set of all T-Fredholm operators is denoted by $F(T)$. The *cohomology* of T is defined by $H[T] = H[\mathfrak{A}/\mathfrak{R}]$, which is an algebraic (and hence unitary) invariant.

Royden's theorem implies

Theorem 7. *The cohomology $H[T]$ of an operator T is isomorphic to the first Čech cohomology group $H^1(\pi_n(T), Z)$, which is also isomorphic to $H[F(T)]$*:

(9) $$H[T] = H[F(T)] = H^1(\pi_n(T), Z).$$

For a T-Fredholm operator A, the *index* $i(A)$ is defined by

(10) $$i(A) = [A^\pi / |A^\pi|],$$

where $[f]$ for a unimodular continuous function f on $\pi_n(T)$ is the (arcwise connected) component of $U[C(\pi_n(T)]$ containing f.

In (10), every step of the mapping:

$$A \to A^\pi \to A^\pi / |A^\pi| \to [A^\pi | A^\pi|]$$

is multiplicative, and the right-hand side of (10) is an element of $H[T]$ by (5). If the composition of the cohomology $H[T]$ is written additively, then the following theorem on elementary properties of the index is obvious; compare with [2; § 4]:

Theorem 8. *The index $i(A)$ on the set $F(T)$ of T-Fredholm operators satisfies*:

(11) $$i(AB) = i(A) + i(B),$$

(12) $$i(A^*) = -i(A),$$

(13) $$i(1) = 0,$$

and

(14) $$i(A+K) = i(A),$$

for every $A, B \in F(T)$ and $K \in \mathfrak{R}$.

In the present general setting, it is uncertain that the index coincides with the usual index due to Atkinson. A special case is discussed in the next section.

4. Example. Let $\mathfrak{A}$ be the unital C^*-algebra generated by the unilateral shift T of multiplicity 1 acting on $\mathfrak{H} = l^2$. By the fact that

$1-TT^*$ is a one-dimensional projection, $\mathfrak{A}$ contains the algebra $\mathfrak{C}(l^2)$ of all compact operators on l^2. In [3], Coburn proved that the pseudoradical of $\mathfrak{A}$ is $\mathfrak{C}(l^2)$ and $\mathfrak{A}/\mathfrak{C}(l^2)$ is isometrically isomorphic to $C(S^1)$ or $S^1=\pi_n(T)$ (cf. [7; §4] for another proof).

It is not hard to see that one can calculate $H^1(S^1, Z)$ to show

(15) $$H[T]=Z.$$

However, (15) is given by an another way: By (5), $H[T]$ is the (multiplicative) group of all unimodular continuous functions modulo the principal component which is isomorphic to the first homotopy group $\pi_1(S^1)=[S^1, S^1]=Z$, cf. (1).

Hence (10) gives

(16) $$i(A)=-\deg\frac{A^\pi}{|A^\pi|},$$

where $\deg f$ is the degree of a mapping f which maps S^1 to S^1. By Theorem 8, $i(A)$ satisfies (11)–(14).

By (16), it is easy to deduce that

(17) $$i(T^n)=-n \quad\text{and}\quad i(T^{*m})=m.$$

Hence, by Theorems 7 and 8, each member of $H[F(T)]$ contains one and only one of T^n or T^{*m}.

On the other hand, the *usual index* of a Fredholm operator A is given by

(18) $$\nu(A)=\dim\ker A-\operatorname{codim}\operatorname{ran} A,$$

from which one has

(19) $$\nu(T^n)=-n \quad\text{and}\quad \nu(T^{*m})=m.$$

Since the pseudoradical of $\mathfrak{A}$ is $\mathfrak{C}(l^2)$, a T-Fredholm operator is a Fredholm operator in the usual sense. Hence the following theorem on the index is proved:

Theorem 9. *For a Fredholm operator A included in the unital C^*-algebra generated by the unilateral shift of multiplicity* 1, *the index $i(A)$ is equal to the usual index $\nu(A)$.*

At this end, a trivial application of the index is listed: An operator R is a square root of T if $R^2=T$. It is well-known that the unilateral shift T has no square root. A weak form of this fact follows from the properties of the index without any computation: if $R\in F(T)$ and $R^2=T$, then $2i(R)=i(T)=-1$ by (11) and (17) which contradicts (15); hence there is no square root of the unilateral shift T which is T-Fredholm.

References

[1] M. Breuer: A generalization of Kuiper's theorem to factors of type II_∞. J. Math. Mech., **16**, 917–925 (1967).

[2] M. Breuer and H. O. Cordes: On Banach algebras with σ-symbol. J. Math. Mech., **13**, 313–323 (1964).

[3] L. A. Coburn: The C^*-algebra generated by an isometry. Bull. Amer. Math. Soc., **73**, 722–726 (1967).
[4] L. A. Coburn and A. Lebow: Algebraic theory of Fredholm operators. J. Math. Mech., **15**, 577–584 (1966).
[5] H. O. Cordes: On a generalized Fredholm theory. Crelle's J., **227**, 121–149 (1967).
[6] H. O. Cordes and J. P. Labrousse: The invariance of the index in the metric space of closed operators. J. Math Mech., **12**, 693–716 (1963).
[7] M. Enomoto, M. Fujii, and K. Tamaki: On normal approximate spectrum. Proc. Japan Acad., **48**, 211–215 (1972).
[8] M. Fujii: On normal approximate spectrum. V. Proc. Japan Acad., **49**, 416–419 (1973).
[9] M. Fujii and R. Nakamoto: On normal approximate spectrum. II. Proc. Japan Acad., **48**, 297–301 (1972).
[10] ——: On normal approximate spectrum. IV. Proc. Japan Acad., **49**, 411–415 (1973).
[11] M. Fujii and K. Tamaki: On normal approximate spectrum. III. Proc. Japan Acad., **48**, 389–393 (1972).
[12] I. Kasahara and H. Takai: Approximate propervalues and characters of C^*-algebras. Proc. Japan Acad., **48**, 91–93 (1972).
[13] N. H. Kuiper: The homotopy type of the unitary group of Hilbert space. Topology, **3**, 19–30 (1965).
[14] C. E. Rickart: General Theory of Banach Algebras. D. van Nostrand, Princeton (1960).
[15] H. L. Royden: Function algebras. Bull. Amer. Math. Soc., **69**, 281–289 (1963).