130. On Some Examples of Non-normal Operators. IV

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1. Introduction. Throughout the note, we shall consider a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$ with the spectrum $\sigma(T)$ and the numerical range $W(T)$. Let us denote further that

(1) $$r(T)=\sup\{|\lambda|;\lambda\in\sigma(T)\}$$

and

(2) $$w(T)=\sup\{|\lambda|;\sigma\in W(T)\}.$$

An operator T is called a *normaloid* if $\|T\|=r(T)$ and a *spectraloid* if $r(T)=w(T)$. T is called a *transaloid* if T satisfies that $\|T-\lambda\|=r(T-\lambda)$ for any complex number λ. Clearly, a transaloid is a normaloid, and conversely T is a transaloid if and only if $T-\lambda$ is a normaloid for every λ. T is called a *convexoid* if $\overline{W}(T)=\operatorname{co}\sigma(T)$ where $\overline{W}(T)$ is the closure of $W(T)$ and $\operatorname{co}S$ is the convex hull of a set S in the complex plane. T is called to satisfy (G_1) if

(3) $$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\operatorname{dist}(\lambda,\sigma(T))}$$

for $\lambda\notin\sigma(T)$. A transaloid is a convexoid, and an operator satisfying (G_1) is a convexoid by [4].

In the present note, we shall characterize transaloids in terms of spectral sets and dilations in §§2–3. In §4, we shall discuss some examples of non-normal operators to disprove certain conjectures which naturally arise from [2]. In this note, we shall denote conveniently by D the unit disk of the complex plane.

2. Spectral sets. A (closed) set S in the plane is a *spectral set* for an operator T if

(4) $$\sigma(T)\subset S$$

and

(5) $$\|f(T)\|\leqq\|f\|_S$$

for any rational function f with poles off S, where

$$\|f\|_S=\sup\{|f(\lambda)|;\lambda\in S\},$$

cf. [5] and [7]. The following theorem is fundamental:

Theorem A (von Neumann [7]). *$\{\lambda;|\lambda-\mu|\leqq k\}$ is a spectral set for an operator T if and only if $\|T-\mu\|\leqq k$.*

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We shall begin with a discussion on a characterization of normaloids:

Theorem 1. *An operator T is a normaloid if and only if $r(T)D$ is a spectral set for T.*

Proof. Suppose that T is a normaloid: $\|T\|=r(T)$. It follows from Theorem A that $r(T)D=\|T\|D$ is a spectral set for T.

Conversely, suppose that $r(T)D$ is a spectral set for T. For $p(\lambda)=\lambda$, we have

$$\|T\|=\|p(T)\|\leqq\|p\|_{r(T)D}=r(T)\leqq\|T\|.$$

Hence T is a normaloid.

Since T is a normaloid if and only if $\|T\|=w(T)$, we have

Corollary 2. *An operator T is a normaloid if and only if $w(T)D$ is a spectral set for T.*

By Theorem 1, we shall give a characterization of transaloids in the following theorem:

Theorem 3. *An operator T is a transaloid if and only if any disk containing the spectrum $\sigma(T)$ is a spectral set for T.*

Proof. It is obvious that S is a spectral set for T if and only if $S-\lambda=\{\mu-\lambda\,;\,\mu\in S\}$ is a spectral set for $T-\lambda$. Hence $r(T-\lambda)D$ is a spectral set for $T-\lambda$ if and only if $r(T-\lambda)D+\lambda$ is a spectral set for T. By Theorem 1 it suffices to prove that $r(T-\lambda)D+\lambda$ is a spectral set for T for any λ if and only if any disk containing $\sigma(T)$ is a spectral set for T.

Suppose that $r(T-\lambda)D+\lambda$ is a spectral set for T for any λ and D' is a disk containing $\sigma(T)$ with the center λ'. Since $r(T-\lambda')D+\lambda'\subset D'$, we can conclude that D' is a spectral set for T.

Conversely, suppose that any disk containing $\sigma(T)$ is a spectral set for T. Since $r(T-\lambda)D+\lambda$ is a disk and contains $\sigma(T)$ for any λ, we have that $r(T-\lambda)D+\lambda$ is a spectral set for T.

By Theorem 3 and Corollary 2, we have an another characterization of transaloids:

Theorem 4. *An operator T is a transaloid if and only if any disk containing $\overline{W}(T)$ is a spectral set for T.*

We can prove Theorem 4 completely analogous method to the proof of Theorem 3, replacing r and σ by w and $\overline{W}$ respectively.

Theorem 4 is a consequence of Theorem 3 by the fact that a disk contains $\sigma(T)$ if and only if it contains $W(T)$.

An operator T is called *numeroid* if $\overline{W}(T)$ is a spectral set for T. By Theorem 4, we have a simple proof of the following.

Theorem 5 (Hildebrandt [4]). *A numeroid is a transaloid.*

Proof. If T is a numeroid, then $\overline{W}(T)$ is a spectral set, so that a disk containing $W(T)$ is a spectral set; hence T is a transaloid by

Theorem 3.

3. Dilations. For an operator T on $\mathfrak{H}$, a normal operator N acting on a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ is called a *normal dilation* of T if N satisfies

$$PNP=TP, \tag{6}$$

where P is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. In addition, if N satisfies

$$PN^nP=T^nP \qquad (n=1,2,\cdots), \tag{7}$$

then N is called a *strong normal dilation* of T. The following theorem on dilations is fundamental in our present study:

Theorem B (Lebow [5]). *Let S be a compact set in the plane and a spectral set for an operator T. Then there exists a strong normal dilation N of T with $\sigma(N)\subset\partial S$ where ∂S is the boundary of S.*

By Theorem B, we have

Lemma 6. *kD is a spectral set for an operator T if and only if there exists a strong normal dilation N of T with $\|N\|=k$.*

Proof. Suppose that kD is a spectral set for T. By Theorem B there exists a strong normal dilation N with $\sigma(N)\subset\partial kD$. Hence we have $\|N\|=k$.

Conversely, suppose that T has a strong normal dilation N with $\|N\|=k$ and f is a rational function with poles off kD. Since $(f(T)x|y)=(f(N)x|y)$ for $x,y\in\mathfrak{H}$, we have

$$\|f(T)\|\leqq\|f(N)\|. \tag{8}$$

Also we have

$$\sigma(N)\subset kD \tag{9}$$

and

$$\|f(N)\|\leqq\|f\|_{\sigma(N)} \tag{10}$$

by the normality of N. It follows from (8), (9) and (10) that kD is a spectral set for T.

The following theorem gives an equivalent condition to define normaloids:

Theorem 7. *An operator T is a normaloid if and only if there exists a strong normal dilation N of T with $\|N\|=r(T)$.*

Poof. By Theorem 1, T is a normaloid if and only if $r(T)D$ is a spectral set for T. By Lemma 6, $r(T)D$ is a spectral set for T if and only if there exists a strong normal dilation N of T with $\|N\|=r(T)$. Hence we have Theorem 7.

By the definition of transaloids and Theorem 7, we have

Theorem 8. *An operator T is a transaloid if and only if there exist strong normal dilations N_λ of $T-\lambda$ with*

$$\|N_\lambda\|=r(T-\lambda) \text{ or } \|N_\lambda\|=w(T-\lambda)$$

for any λ.

4. Examples. An operator T is a *spectroid* (or *hen-spectroid*, resp.) if $\sigma(T)$ ($\tilde{\sigma}(T)$, the complement of the unbounded component of the complement of $\sigma(T)$, resp.) is a spectral set for T. Since

(11) $$\sigma(T) \subset \tilde{\sigma}(T) \subset \mathrm{co}\,\sigma(T) \subset \overline{W}(T),$$

a spectroid is a hen-spectroid and a hen-spectroid is a numeroid by [2; III].

The above discussion tells us that the classes of all spectroids, hen-spectroids, numeroids, transaloids and normaloids are determined by certain spectral sets or by strong normal dilations with specific spectral properties. Moreover, up to numeroids, these classes are characterized by the functions of operators: The class of all transaloids is determined by the fact that $T-\lambda$ is a normaloid for any λ. The other two classes are characterized as follows:

Theorem C (Berberian [1]). *T is a spectroid if and only if $f(T)$ is a normaloid for every rational functions with poles off $\sigma(T)$.*

Theorem D ([2; III, Theorem 13]). *T is a hen-spectroid if and only if $p(T)$ is a normaloid for every polynomial p.*

According to a closed analogy between the classes of operators defined by spectral sets and growth conditions, a question naturally arise: Is it possible that an operator satisfying (G_1) is characterized by its functions? Unfortunately, the conjecture is negatively answered in the following

Theorem 9. *There is an operator T satisfying (G_1) such that $T^2-\lambda$ is not a spectraloid for some λ.*

Proof. Our example is same with that of Luecke [6; Theorem 4]: Suppose

$$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If N is a normal operator with $\sigma(N)=W(A)$, then $T=A\oplus N$ satisfies (G_1) by Luecke's theorem. Clearly $T^2=A^2\oplus N^2$. Since $\sigma(N)\subset\frac{1}{2}D+1$ and $\mathrm{co}\,\sigma(T^2)=\mathrm{co}\,\sigma(T)^2$, we see that $\mathrm{co}\,\sigma(T^2)$ is contained in the open right half plane. On the other hand, $W(A^2)=D+1$, so that we have

$$0\in\overline{W}(T^2)-\mathrm{co}\,\sigma(T^2).$$

Therefore, T^2 is not a convexoid.

By a characterization theorem for convexoids in [3; Theorem 3], if $T^2-\lambda$ is a spectraloid for every λ, then T^2 is a convexoid. Hence there is some such that $T^2-\lambda$ is not a spectraloid by the above.

The following theorem disproves the converse direction of our conjecture:

Theorem 9′. *There is an operator T not satisfying (G_1) such that $p(T)$ is a spectraloid for every polynomial p.*

Proof. The example is same with [2; III]: If

$$B=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and U is the simple bilateral shift, then $T=B\oplus U$ is a hen-spectroid as in the proof of [2; III, Theorem 8]. By [2, III, Theorem 13], $p(T)$ is a normaloid and consequently a spectraloid for every polynomial p. On the other hand, T does not satisfy (G_1) by [2; III, Theorem 8].

But, T in the proof of Theorem 9′ satisfies (H_1) in the sense that

$$\|(T-\lambda)^{-1}\| \leqq \frac{1}{\operatorname{dist}(\lambda, \tilde{\sigma}(T))} \tag{12}$$

for every $\lambda \notin \tilde{\sigma}(T)$, since a hen-spectroid satisfies (H_1) by [2; III, Proposition 3]. Therefore, we can still hope: Is it true that T satisfies (H_1) if $p(T)$ is a spectraloid for every polynomial p? In our present knowledge, we have no evidence to decide our conjecture. However, we can prove the following weaker statement:

Theorem 10. *There is an operator T such that T does not satisfy (G_1) or (H_1) and T^n is a convexoid for every $n=1,2,\cdots$.*

Proof. Let

$$T=\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the mutually distinct cubic roots of 1. By [2; I, Theorem 2], T is a convexoid. Clearly, T^n is normal for $n \geqq 2$. Hence T satisfies the second requirement of the theorem. However, T does not satisfy (H_1) as already observed in [2; II, Proposition 10].

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