153. On Exceptional Linear Combinations of Entire Functions

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1. Introduction.

As an interesting result with respect to the relations between the sum of deficiencies and the number of Picard's exceptional values of entire algebroid functions, K. Niino and M. Ozawa [3], M. Ozawa [5] and T. Suzuki [6] showed the following fact: Let $f(z)$ be a transcendental entire algebroid function defined by an irreducible equation

$$F(z, f) \equiv f^n + A_1(z) f^{n-1} + \cdots + A_n(z) = 0,$$

where $A_1, \cdots, A_n$ are entire functions and $n=3, 4, 5$. Let $\{a_j\}_{j=0}^n$ be distinct finite numbers such that arbitrary $n-1$ functions of $\{F(z, a_j)\}_{j=0}^n$ are linearly independent and

$$\sum_{j=0}^{n} \delta(a_j, f) + \sum_{\nu=1}^{n-3} \delta(a_{j_\nu}, f) > 2n-3$$

for all $n-3$ numbers $\{a_{j_\nu}\}_{\nu=1}^{n-3}$ of $\{a_j\}_{j=0}^n$. Then there exists at least one Picard's exceptional value in $\{a_j\}_{j=0}^n$. Moreover J. Noguchi [4] showed that this result is available for all $n \geqq 2$ and in the case of $n=5$, he obtained a better result.

In this note, we will discuss the case of transcendental system of entire functions and give an extension of the above fact. In the proof of Theorem 1, methods of J. Noguchi are used.

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2. Preliminaries.

Let $f_0, \cdots, f_l$ be entire functions and $X = \{F_i\}_{i=0}^N$ $(l \leqq N \leqq \infty)$ a set of linear combinations of $f_0, \cdots, f_l$ with constant coefficients. We say that X is a regular family of linear combinations of $f_0, \cdots, f_l$ when the matrices of the coefficients $(a_{i_n j})_{j=0,\cdots,l}^{n=0,\cdots,l}$ are regular for all $l+1$ integers $\{i_n\}_{n=0}^l$ $(0 \leqq i_n \leqq N)$. And we say that the elements $\{G_k\}_{k=1}^p$ in X form a basis of X if and only if $G_1, \cdots, G_p$ are linearly independent and all of X can be represented as linear combinations of $G_1, \cdots, G_p$.

Let $f = (f_0, \cdots, f_n)$ $(n \geqq 1)$ be a transcendental system in $|z| < \infty$. Namely $f_0, \cdots, f_n$ are entire functions without common zero and $\lim_{r\to\infty} T(r, f)/\log r = \infty$, where $T(r, f)$ is the characteristic function of f defined by Cartan [1], i.e. $u(re^{i\theta}) = \max_{0\leqq j\leqq n} \log|f_j(re^{i\theta})|$ and $T(r, f)$

$=\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta-u(0)$. Moreover the deficiency of linear combination F is defined by $\delta(F)=1-\limsup_{r\to\infty}\frac{N(r,0,F)}{T(r,f)}$.

Lemma 1. *Let $X=\{F\}$ be a regular family of linear combinations of $f_0,\cdots,f_n$ which are linearly independent, then we have*

$$\sum_{F\in X}\delta(F)\leqq n+1$$

(*Cartan* [1]).

Lemma 2. *Let $X=\{F_i\}_{i=0}^N$ be a regular family of linear combinations of $f_0,\cdots,f_n$ and $\{G_k\}_{k=1}^l$ $(l\leqq N+1\leqq\infty)$ a basis of X. Then we have*

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leqq k\leqq l}\{\log|G_k(re^{i\theta})|\}d\theta+O(1).$$

This result follows at once from the definitions of regular family and $T(r,f)$.

Lemma 3. *Let $F_1,\cdots,F_l$ be linearly independent entire functions in X and put $F=F_1+\cdots+F_l$. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leqq j\leqq l}\{\log^+|F_j(re^{i\theta})|\}d\theta\leqq\sum_{j=1}^{l}N(r,0,F_j)+m(r,F)+S(r),$$

where $S(r)=O(\log T(r,f)+\log r)$ as $r\to\infty$ possibly outside a set of r of finite linear measure when the order of $T(r,f)$ is infinite (*Nevanlinna* [2]).

Proof. By $F_1+\cdots+F_l=F$ and $F_1^{(\mu)}+\cdots+F_l^{(\mu)}=F^{(\mu)}$ for $\mu\geqq 1$, we have

$$F_j=\frac{\Delta_j}{\Delta},\qquad j=1,\cdots,l,$$

where, using the Wronskian $W(F_1,\cdots,F_l)$ of $F_1,\cdots,F_l$,

$$\Delta=W(F_1,\cdots,F_l)/F_1\cdots F_l\qquad(\neq 0)$$

and

$$\Delta_j=W(F_1,\cdots,F_{j-1},F,F_{j+1},\cdots,F_l)/F_1\cdots F_{j-1}F_{j+1}\cdots F_l,\qquad j=1,\cdots,l.$$

Put $\Delta_j=F\tilde{\Delta}_j$, then we have

$$\max_{1\leqq j\leqq l}\{\log^+|F_j|\}\leqq\max_{1\leqq j\leqq l}\left\{\log^+|F|+\log^+|\tilde{\Delta}_j|+\log^+\left|\frac{1}{\Delta}\right|\right\}$$
$$\leqq\sum_{j=1}^{l}\log^+|\tilde{\Delta}_j|+\log^+|F|+\log^+\left|\frac{1}{\Delta}\right|.$$

Hence,

$$\frac{1}{2\pi}\int_0^{2\pi}\max_{1\leqq j\leqq l}\{\log^+|F_j(re^{i\theta})|\}d\theta\leqq\sum_{j=1}^{l}m(r,\tilde{\Delta}_j)+m(r,F)+T\left(r,\frac{1}{\Delta}\right)$$
$$=\sum_{j=1}^{l}m(r,\tilde{\Delta}_j)+m(r,F)+m(r,\Delta)+N(r,\Delta)+O(1).$$

On the other hand, we see $N(r, \Delta)\leqq\sum_{j=1}^{l} N(r, 0, F_j)$ because $F_1\cdots F_l\Delta$ is entire. In the same method used by Cartan to estimate an error term in the proof of the fundamental theorem ([1], p. 12–p. 15), we have $\sum_{j=1}^{l} m(r, \tilde{\Delta}_j)+m(r, \Delta)=S(r)$. Thus we have the desired result.

3. Exceptional linear combinations.

Theorem 1. *Let $F, F_0, \cdots, F_n$ be a regular family of linear combinations of $f_0, \cdots, f_n$ and $F_0, \cdots, F_n$ satisfy the following conditions:*

(i) *Arbitrary $n-1$ functions in $\{F_j\}_{j=0}^n$ are linearly independent.*

(ii) $\sum_{j=0}^{n} \delta(F_j)+\sum_{\nu=1}^{n-3} \delta(F_{j_\nu})>2n-3+2\xi$

for all $n-3$ functions $\{F_{j_\nu}\}_{\nu=1}^{n-3}$ in $\{F_j\}_{j=0}^n$.
Then there exists a function F_{j_0} in $\{F_j\}_{j=0}^n$ such that $\alpha F_{j_0}=F$, where α is non-zero constant and

$$\xi=\limsup_{r\to\infty} \frac{m(r, F)}{T(r, f)} \qquad \left(0\leqq\xi<\frac{1}{2}\right).$$

Proof. Let λ be the number of distinct non-trivial linear relations among $f_0, \cdots, f_n$. Then condition (i) implies $0\leqq\lambda\leqq 2$. We shall show that in this case λ is equal to 1 in the following.

If $\lambda=0$, then $F_0, \cdots, F_n$ are linearly independent. By the definition of regular family, we can see easily

$$a_0F_0+\cdots+a_nF_n=F, \qquad a_j\neq 0\ (j=0, \cdots, n).$$

So we have by Lemma 2 and Lemma 3,

$$T(r, f)\leqq\sum_{j=0}^{n} N(r, 0, F_j)+m(r, F)+S(r).$$

Hence

$$\sum_{j=0}^{n} \delta(F_j)\leqq n+\xi.$$

This is a contradiction.

If $\lambda=2$, there exist $n-1$ functions in $\{F_j\}_{j=0}^n$ that form a basis of $\{F_j\}_{j=0}^n$. Let, for example, $\{F_0, \cdots, F_{n-2}\}$ be the basis, then each of $\{F_0, \cdots, F_{n-2}, F_{n-1}\}$ and $\{F_0, \cdots, F_{n-2}, F_n\}$ is a regular family of linear combinations of $\{F_0, \cdots, F_{n-2}\}$ because of the condition (i). By Lemma 1,

$$\sum_{j=0}^{n-1} \delta(F_j)\leqq n-1.$$

This leads also to a contradiction. Thus we have $\lambda=1$.

Suppose that any n functions in $\{F_j\}_{j=0}^n$ are linearly independent. Since $\lambda=1$, there exist n functions in $\{F_j\}_{j=0}^n$ that form a basis of $\{F_j\}_{j=0}^n$. Let, for example, $F_{i_0}, \cdots, F_{i_{n-1}}$ be the basis, we can write

$$F_{i_n}=a_0F_{i_0}+\cdots+a_{n-1}F_{i_{n-1}}, \qquad a_j\neq 0\ (j=0, \cdots, n-1)$$

by our assumption. Hence $\{F_j\}_{j=0}^n$ is a regular family of linear combination of $F_{i_0}, \cdots, F_{i_{n-1}}$. So similarly in the above, we have

$$\sum_{j=0}^{n} \delta(F_j) \leqq n.$$

This is a contradiction.

Now we may assume that $F_0, \cdots, F_{n-1}$ are linearly dependent;

$$(1) \qquad \sum_{j=0}^{n-1} \beta_j F_j = 0, \qquad \beta_j \neq 0 \ (j=0, \cdots, n-1)$$

by the condition (i). Since $\lambda=1$, n functions of $\{F_j\}_{j=0}^{n}$ one of which is F_n, are linearly independent and form a basis. By our assumption of regular family, we have

$$(2) \qquad a_0 F_0 + \cdots + a_n F_n = F, \qquad a_j \neq 0 \ (j=0, \cdots, n).$$

Set $\beta_0 = a_0$ and from (1) and (2), we obtain

$$(a_1-\beta_1)F_1 + \cdots + (a_{n-1}-\beta_{n-1})F_{n-1} + a_n F_n = F.$$

Hence we have

$$(3) \qquad \alpha_1 F_1 + \cdots + \alpha_n F_n = F, \qquad \alpha_n \neq 0.$$

If all $\alpha_j \neq 0$, since $F_1, \cdots, F_n$ form a basis of $\{F_j\}_{j=0}^{n}$, using Lemma 2 and Lemma 3, we obtain

$$\sum_{j=1}^{n} \delta(F_j) \leqq n-1+\xi,$$

and this is a contradiction. Thus we may set $\alpha_1=0$. Moreover we will show that $\alpha_2, \cdots, \alpha_{n-1}$ are zero.

Assume that non-zero elements of $\{\alpha_2, \cdots, \alpha_{n-1}\}$ are $\alpha_k, \cdots \alpha_{n-1}$, $2 \leqq k \leqq n-1$. The equation (3) is reduced to

$$(4) \qquad \alpha_k F_k + \cdots + \alpha_n F_n = F \qquad (\alpha_j \neq 0).$$

Set $\beta_k = \alpha_k$ and from (1) and (4), we have

$$-\beta_0 F_0 - \cdots - \beta_{k-1} F_{k-1} + (\alpha_{k+1}-\beta_{k+1})F_{k+1} + \cdots + (\alpha_{n-1}-\beta_{n-1})F_{n-1} + \alpha_n F_n = F.$$

Since $F_0, \cdots, F_{k-1}, F_{k+1}, \cdots, F_n$ form a basis of $\{F_j\}_{j=0}^{n}$, one of their coefficients is zero by Lemma 2 and Lemma 3 similarly.

Let $\alpha_{k+1}-\beta_{k+1}=0$ and we have

$$(5) \qquad -\beta_0 F_0 - \cdots - \beta_{k-1} F_{k-1} + (\alpha_{k+2}-\beta_{k+2})F_{k+2} + \cdots + (\alpha_{n-1}-\beta_{n-1})F_{n-1} + \alpha_n F_n = F.$$

Let $\{F_{j_\nu}\}_{\nu=1}^{l}$ be the functions of $\{F_j\}_{j=0}^{n}$ which appear with non-zero coefficients in both equations (4) and (5). Then $1 \leqq l \leqq n-k-1 \leqq n-3$. Applying Lemma 2 and Lemma 3 to the equations (4) and (5), we have

$$T(r, f) \leqq \sum_{j=0}^{n} N(r, 0, F_j) + \sum_{\nu=1}^{l} N(r, 0, F_{j_\nu}) + 2m(r, F) + S(r),$$

so that

$$\sum_{j=0}^{n} \delta(F_j) + \sum_{\nu=1}^{l} \delta(F_{j_\nu}) \leqq n+l+2\xi.$$

Let $\{F_{j_\nu}\}_{\nu=l+1}^{n-3}$ be any $n-l-3$ numbers of $\{F_j\}_{j=0}^{n} - \{F_{j_\nu}\}_{\nu=1}^{l}$, then

$$\sum_{j=0}^{n} \delta(F_j) + \sum_{\nu=1}^{n-3} \delta(F_{j_\nu}) \leqq 2n-3+2\xi.$$

This is a contradiction. That is to say we have $\alpha_n F_n = F$. Thus Theorem 1 follows.

Remark. If F is a exceptional linear combination in the sense of Picard (resp. lacunary), then F_{j_0} is also exceptional linear combination in the sense of Picard (resp. lacunary).

If especially $F\equiv 1$, then we obtain the result of J. Noguchi in the introduction.

In the case of $n=5$, we obtain a slightly better following theorem

Theorem 2. *Let $F, F_0, \cdots, F_5$ be a regular family of linear combinations of $f_0, \cdots f_5$ and $F_0, \cdots, F_5$ satisfy the following conditions:*

(i) *Arbitrary four functions of $\{F\}_{j=0}^5$ are linearly independent.*

(ii) $\sum_{j=0}^{5} \delta(F_j)+\delta(F_k)>6+2\xi$ *for all F_k.*

Then there exists a function F_{j_0} in $\{F\}_{j=0}^5$ such that $\alpha F_{j_0}=F$, where α is non-zero constant and ξ is the same value defined in Theorem 1.

We obtain easily the above result by considering Theorem 1 in relation to Theorem 2 in [4].

References

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