169. *Theorems on the Finite-dimensionality of Cohomology Groups.* V

By Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University

(Comm. by Kôsaku YOSIDA, M. J. A., Dec. 12, 1973)

The purpose of this note is to present some theorems which assert the stability of cohomolgy groups attached to an elliptic complex $\mathcal{M}$ under the deformation of the domain. The most essential step in our argument is Lemma 1, which seems to be of its own interest.

We use the same notations and terminologies as in our previous notes [2], [3] and do not repeat their definitions. All problems in this note are considered in the real analytic category, that is, the manifold under consideration is real analytic, the (pseudo-) differential operators considered here have real analytic coefficients and so on.

The present writer expresses his heartiest thanks to Professor D.C. Spencer and Mr. M. Kashiwara for many valuable discussions with them, from which stemmed the investigation of the problems discussed in this note. Further details of this note will appear somewhere else.

Throughout this note (except in the last remark) we always assume the following:

(1) $\mathcal{M}$ denotes an elliptic system of linear differential equations defined on M, which admits a free resolution of length d by the sheaf $\mathcal{D}^f$ of linear differential operators of finite order on M.

(2) $\mathcal{M}$ is purely d-dimensional, i.e., $\mathcal{E}xt^j_{\mathcal{D}^f}(\mathcal{M}, \mathcal{D}^f)=0,\ j\neq d$.

In the sequel we denote by $\mathcal{M}'$ the adjoint system of $\mathcal{M}$. The system $\mathcal{M}'$ is by definition the left $\mathcal{D}^f$-Module given by $\mathcal{E}xt^d_{\mathcal{D}^f}(\mathcal{M}, \mathcal{D}^f)\otimes_{\mathcal{O}}(\Omega^n)^{\otimes -1}$ where Ω^n is the sheaf of holomorphic n-forms.

In this note we essentially use the notion of the "negative" tangential system $(\mathcal{N}')_-$ (of pseudo-differential equations) of $\mathcal{M}'$ defined (on the cotangential sphere bundle of the boundary) via the "negative characteristics" of $\mathcal{M}'$. The "negative" tangential system of $\mathcal{M}'$ can be defined in an analogous way to the "positive" tangential system $\mathcal{N}_+$ of $\mathcal{M}$. (Cf. Kashiwara-Kawai [1], Kawai [2].)

Now we have the following lemma, which supplements Lemma 2 of Kawai [3]. (See also Kuranishi [5] for some related topics.)

Lemma 1. *Let $\Omega=\{x\in M; \varphi(x)<0\}$ be a relatively compact domain with C^∞-boundary. Assume that the system $\mathcal{N}_+$ is either $(q+1)$-convex or $(q-1)$-concave at any point in its real characteristic variety.*

Further assume that $(\mathcal{N}')_-$ is either $(d-q)$-convex or $(d-q-2)$-concave at any point in its real characteristic variety. Then we have the following isomorphism:

$$\mathrm{Ext}^q_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Ext}^q_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}). \tag{3}$$

In order to prove this lemma we apply the results of Sato-Kawai-Kashiwara [6] Chapter III §2.3 and those of Kashiwara-Kawai [1] to study the cohomology groups with compact support attached to the adjoint complex $\mathcal{M}'$. We also use Serre's duality theorem (see Komatsu [4] for example) to obtain the above result from the result for the adjoint complex. Note that Theorem 1 of Kawai [2] tells us that the cohomology group in the above is finite-dimensional.

Remark. Since $\mathcal{M}$ is a system of linear *differential* (not pseudo-differential) equations, the condition on $(\mathcal{N}')_-$ in the above follows from the condition on $\mathcal{N}_+$. However, we have presented Lemma 1 in the above form because we think it better to call the reader's attention to the importance of $(\mathcal{N}')_-$ in showing the "continuity property" of the cohomology groups. Taking account of this remark we state the theorems using only the "positive" tangential systems of $\mathcal{M}$ in the below.

Making full use of Lemma 1 we obtain the following

Theorem 2. *Let Ω be a connected open set in M. Assume that there exists a real valued real analytic function $\varphi(x)$ defined on Ω which satisfies the following:*

(4) $\Omega_t = \{x \in \Omega; \varphi(x) < t\}$ *is relatively compact for any* $t \in \mathbf{R}$.

(5) Ω_t *has C^ω-boundary $\partial\Omega_t$ as long as* $\Omega_t \neq \emptyset$.

Further assume that the "positive" tangential system $\mathcal{N}_{t,+}$ (defined on the cotangential sphere bundle of $\partial\Omega_t$) of $\mathcal{M}$ is $(q+1)$-convex or $(q-1)$-concave at any point in its real characteristic variety. Then we have the following isomorphism

$$\mathrm{Ext}^q_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}) \xrightarrow{\sim} \varprojlim_{t\to\infty} \mathrm{Ext}^q_{\mathcal{D}}(\Omega_t; \mathcal{M}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Ext}^q_{\mathcal{D}}(\Omega_t; \mathcal{M}, \mathcal{B}) \tag{6}$$

as long as $\Omega_t \neq \emptyset$.

Corollary. *Assume that $K_0 = \{x \in \Omega; \varphi(x) \leqq 0\}$ reduces to a single point x_0 and that $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M}, \mathcal{A}) = 0$ holds near x_0 for $1 \leqq j \leqq q$. Then the same hypotheses as in Theorem 2 imply that*

$$\mathrm{Ext}^q_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}) = 0 \tag{7}$$

holds.

Remark. We hope that Theorem 2 is one of the best possible results of this sort from the analytical view point, though we have assumed there some severe conditions (1) and (2) from the algebraic view point. On the other hand main results in Kawai [2], [3] hold without such algebraic conditions. For example we can prove Theorem

1 in Kawai [2] and Theorem 3 and Theorem 4 in Kawai [3] only assuming the existence of local free resolution of $\mathcal{M}$ if we use the cohomology groups of coverings. The present writer owes the idea of such an argument to Mr. Kashiwara. Note that the local existence of free resolution of $\mathcal{M}$ is assured by the Spencer sequence. (See for example Spencer [7].)

References

[1] Kashiwara, M., and T. Kawai: On the boundary value problem for elliptic system of linear differential equations. I. Proc. Japan Acad., **48**, 712–715 (1972).

[2] Kawai, T.: Theorems on the finite-dimensionality of cohomology groups. III. Proc. Japan Acad., **49**, 243–246 (1973).

[3] ——: Ibid. IV. Proc. Japan Acad., **49**, 655–658 (1973).

[4] Komatsu, H.: Projective and injective limits of weakly compact sequences of locally convex spaces. J. Math. Soc. Japan, **19**, 366–383 (1967).

[5] Kuranishi, M.: On the complexes on the boundary induced by elliptic complexes of differential operators (to appear).

[6] Sato, M., T. Kawai, and M. Kashiwara: Microfunctions and Pseudo-Differential Equations. Lecture Note in Mathematics No. 287, Springer, Berlin-Heidelberg-New York, pp. 265–529 (1973).

[7] Spencer, D. C.: Overdetermined systems of linear partial differential equations. Bull. A. M. S., **75**, 179–239 (1969).