7. *On a Relation between Characters of Discrete and Non-Unitary Principal Series Representations*

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§ 1. **Introduction.** For the general linear group $G=SL(2,R)$, it was proved by I. M. Gelfand and M. I. Graev, N. Ya Vilenkin in [6] that the quotient representation of certain non-unitary principal series representations by its finite dimentional invariant subrepresentation is infinitesimaly equivalent to a representation which belongs to the discrete series.

Our purpose is to prove a similar relation for any group G satisfying the following conditions:

(C.1) G is a connected real simple Lie group.

(C.2) There is a simply connected complex simple Lie group G_c which is the complexification of G.

(C.3) The symmetric space G/K is of rank one and G has a compact Cartan subgroup, where K denotes the maximal compact subgroup of G.

In § 3, we prove the relation using the explicit character formulas for the representations in discrete series and in non-unitary principal series obtained by Harish-Chandra ([2], [4], [5]).

In § 4, we state some results for $G=\text{Spin}\,(2l,1)$ $(l\geqq 1)$ using Theorem 1.

§ 2. **Preliminaries.** Let G be a Lie group satisfying conditions C.1, C.2 and C. 3 with Lie algebra $\mathfrak{g}$. We shall always denote by $\mathfrak{L}_c$ the complexification of Lie sub-algebra $\mathfrak{L}$ of $\mathfrak{g}$. By C.2, $\mathfrak{g}_c$ is the Lie algebra of G_c.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition and K be the analytic subgroup of G whose Lie algebra is $\mathfrak{k}$. We shall fix a Cartan subalgebra $\mathfrak{b}(\subset\mathfrak{k})$ of $\mathfrak{g}$. Let Ω be the non-zero root system of $\mathfrak{g}_c$ with respect to $\mathfrak{b}_c$. For any root α, we can select a root vector X_α such that $B(X_\alpha,X_{-\alpha})=1$ (Where B is the Killing form of $\mathfrak{g}_c$). As usual we identify $\mathfrak{b}_c$ with the dual space of $\mathfrak{b}_c$ by the relation $\lambda(H)=B(H,H_\lambda)$ and denote $(\lambda,\mu)=B(H_\lambda,H_\mu)$ for two linear functions λ,μ on $\mathfrak{b}_c$. Then we have $[X_\alpha,X_{-\alpha}]=H_\alpha$ for any root $\alpha\in\Omega$. For a fixed non-compact root γ, we select a compatible ordering in dual space of RH_γ and $\sqrt{-1}b$ such that $\gamma>0$. Put

$$y=\exp\left\{\frac{\sqrt{-1}\pi}{4}\cdot 2^{1/2}((\gamma,\gamma))^{-1/2}(X_\gamma-X_{-\gamma})\right\}\in G_c.$$

Then $Ad(y^{-1})\mathfrak{b}_c=\mathfrak{a}_c$ where $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{a}_R=R\sqrt{-1}(X_\gamma+X_{-\gamma})$ and $\mathfrak{a}_I=\mathfrak{a}\cap\mathfrak{k}$. Then $=\mathfrak{a}_R+\mathfrak{a}_I$ and $\{\mathfrak{a},\mathfrak{b}\}$ is a complete set of representatives of non-conjugate Cartan subalgebras in $\mathfrak{g}$. Since $\mathfrak{b}_c=Ad(y)\mathfrak{a}_c$, for any linear function λ on $\mathfrak{b}_c$, we can define a linear function λ^y on $\mathfrak{a}_c$ as follows;

$$\lambda^y(H)=\lambda(Ad(y)H)\qquad\text{for all } H\in\mathfrak{a}_c.$$

In this way $\Omega^y=\{\alpha^y\mid\alpha\in\Omega\}$ is the non zero root system of $\mathfrak{g}_c$ with respect to $\mathfrak{a}_c$. The ordering of Ω induces a lexicographic order in Ω^y.

For any root $\alpha\in\Omega^y$, put $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}_c\mid ad(H)X=\alpha(H)X \text{ for all } H\in\mathfrak{a}_c\}$. Then $\mathfrak{g}_c=\mathfrak{a}_c+\sum_{\alpha\in\Omega^y}\mathfrak{g}_\alpha$.

Put $\mathfrak{n}=\mathfrak{g}\cap\sum_{\alpha\in\Omega^y,(\alpha,\gamma^y)>0}\mathfrak{g}_\alpha)$ and let A_R and N be the analytic subgroups of G corresponding to $\mathfrak{a}_R$ and $\mathfrak{n}$. Then $G=KA_RN$. Define the functionals ρ_+,ρ_- on $\mathfrak{a}_c$ as follows:

$$\rho_+=\frac{1}{2}\sum_{\alpha\in\Omega^y,\alpha>0,(\alpha,\gamma^y)\neq 0}\alpha,\qquad \rho_-=\frac{1}{2}\sum_{\alpha\in\Omega^y,\alpha>0(\alpha,\gamma^y)=0}\alpha.$$

And define the functional ρ on $\mathfrak{b}_c$ by $\rho=\frac{1}{2}\sum_{\alpha\in\Omega,\alpha>0}\alpha$.

§ 3. **Main result.** Let $dk(k\in K)$ be the Haar measure of K normalized as $\int_K dk=1$. And let $L_2(K)$ be the set of all square integrable functions on K with respect to dk. For any $x\in G$ and any $k\in K$, define $H(x,k)(\in\mathfrak{a}_R)$, $k_x(\in K)$ as follows:

$$xk\in k_x\exp H(x,k)N,\quad k_x\in K,\quad \exp H(x,k)\in A_R.$$

Let M be the centralizer of $\mathfrak{a}_R$ in K. Then M is compact. Let σ be an irreducible unitary representation of M and μ be a linear function on $\mathfrak{a}_R$. Put $L_2^\sigma(K)$ by

$$L_2^\sigma(K)=\{\phi\in L_2(K)\mid\phi(mk)=\sigma(m)\phi(k)\}.$$

Define the representation $T^{\sigma,\mu}$ of G as follows:

$$[T^{\sigma,\mu}(x)\phi](k)=e^{-(\mu+\rho_+)(H(x^{-1},k))}\phi(kx^{-1}),$$

for all $x\in G$ and all $\phi=\phi(k)\in L_2^\sigma(K)$.

Then the trace of $T^{\sigma,\mu}$ defines a distribution on G (see [2]).

We shall denote this distribution by trace $T^{\sigma,\mu}$.

Let $W(W_t)$ be the Weyl group of $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) with respect to b_c. Put $W_0=\{s\in W\mid s\mathfrak{a}_I=\mathfrak{a}_I\}$. Then W_0 is a subgroup of W. Put $\Omega_0=\{\alpha\in\Omega\mid\alpha=\gamma$ or α is positive such that $(\alpha,\gamma)=0\}$. Define the subset W_1 of W by $W_1=\{s\in W\mid s\alpha>0$ for all $\alpha\in\Omega_0\}$.

For any dominant integral form λ on $\mathfrak{b}_c$ and any $s\in W_1$, define the linear form $\mu=\mu(s,\lambda)$ on $\mathfrak{a}_c$ and the irreducible representation $\sigma(s,\lambda)$ of M as follows:

$\mu(s,\lambda)(H)=(s(\lambda+\rho))^{y}(H)$ for all $H\in\mathfrak{a}_R$,

$\sigma(s,\lambda)=$ the irreducible representation of M with the highest weight $(s(\lambda+\rho))^{y}-\rho_-|\mathfrak{a}_I$, where $(s(\lambda+\rho))^{y}-\rho_-|\mathfrak{a}_I$ is the restriction of linear form $(s(\lambda+\rho))^{y}-\rho_-$ on $\mathfrak{a}$ to $\mathfrak{a}_I$.

Define the representation $V_{s(\lambda+\rho)}$ of G by

$$V_{s(\lambda+\rho)}(x)=T^{\sigma(s,\lambda),\mu(s,\lambda)}(x), \qquad (x\in G).$$

In the following, we denote by λ a dominant integral form on $\mathfrak{b}_c$. Let $\pi_{\lambda+\rho}$ be the finite dimensional irreducible representation of G with the highest weight λ. Then trace $\pi_{\lambda+\rho}$ defines a distribution on G by

$$[\text{trace }(\pi_{\lambda+\rho})](f)=\int_G \text{trace }\pi_{\lambda+\rho}(x)f(x)dx$$

for any $f\in C_c^\infty(G)$, where $C_c^\infty(G)$ is the set of all C^∞-functions on G with compact supports, and $dx(x\in G)$ is a Haar measure on G. Let $\Theta_{s(\lambda+\rho)}(s\in W)$ be the Harish-Chandra's character for discrete series [5]. Then we have the following theorem.

Theorem 1. *Let* $\Theta^*_{\lambda+\rho}=\sum_{s\in W_{\mathfrak{k}}\backslash W}\Theta_{s(\lambda+\rho)}$. *Then we have*

$$\Theta^*_{\lambda+\rho}=(-1)^q\left\{\text{trace }\pi_{\lambda+\rho}-\sum_{s\in W_1}\varepsilon(s)\text{ trace }[V_{s(\lambda+\rho)}]\right\},$$

where $q=\frac{1}{2}\dim G/K$.

Our proof of this theorem is obtained from the explicit formulas of characters $\Theta_{s(\lambda+\rho)}$ ([2]–[5]) and trace $V_{s'(\lambda+\rho)}(s'\in W_1)$([2]).

§ 4. **An application.** Let $\mathcal{E}_K$ be the set of all equivalence classes of irreducible representations of K. For any representation π of K, we denote the multiplicity of δ in π by $[\pi;\delta]$ $(\delta\in\mathcal{E}_K)$. And by $\tau|K$, we mean the restriction of a representation r of G to K. For any $f\in C_c^\infty(G)$, we define the function f^δ by $f^\delta(x)=\bar{\chi}_\delta * f * \bar{\chi}_\delta(x)(x\in G)$ where $*$ is the convolution on K and $\chi_\delta=\deg(\delta)$ trace (δ).

In this section, we shall assume that $G=\text{Spin}(2l,1)(l\geqq 1)$. Let $P_{\mathfrak{p}}$ be the set of all non-compact positive roots in Ω. Then $P_{\mathfrak{p}}=\{\lambda_1,\lambda_2,\cdots,\lambda_l\}$ $(l=\dim\mathfrak{b})$, where λ'_is are linear forms which are mutually orthogonal with respect to the Killing form B. And the set $P_{\mathfrak{k}}$ of all compact positive roots is

$$\{\lambda_i\pm\lambda_j|1\leqq i\leqq j\leqq l\}.$$

Let λ be a dominant integral form on $\mathfrak{b}_c$. Then $=m_1\lambda_1+m_2\lambda_2+\cdots+m_l\lambda_l$, $m_1\geqq m_2\geqq\cdots\geqq m_l\geqq 0$, and m'_is are either all integers or all strict half integers. Put

$$\mathcal{E}_\lambda=\left\{\eta=\sum_{i=1}^{l}\eta_i\lambda_i|\eta_1\geqq m_1+1\geqq\eta_2\geqq\cdots\geqq\eta_l\geqq m_l+1,\right.$$
$$\left.\eta_i\equiv m_i\ (\text{mod }Z)\qquad i=1,2,\cdots,l\right\}$$

where Z is the set of all integers. Then we have the following formulas for any function $f\in C_c^\infty(G)$.

Theorem 2. 1) *For any irreducible representation* $\delta=\delta_\eta$ *of* K *which has the highest weight* $\eta\in\mathcal{E}_\lambda$.

$$\Theta^*_{\lambda+\rho}(f^\delta)=(\text{trace } V_{s_0(\lambda+\rho)})(f^\delta)$$

where $s_0=s_{\lambda_l-\lambda_{l-1}}s_{\lambda_l-\lambda_{l-2}}\cdots s_{\lambda_l-\lambda_1}(\in W_1)$.

2) *For the representation* $\omega_{\lambda+\rho}$ *corresponding to* $\Theta^*_{\lambda+\rho}$

$$[\omega_{\lambda+\rho}|K;\delta]=1 \qquad \textit{for all } \delta=\delta_\eta(\eta\in\mathcal{E}_\lambda).$$

Remark. This result is known (T. Hirai [7], [8]). But we shall prove it by a different method from his. For the proof of Theorem 2, we shall state two lemmas.

Lemma 1. *Let* $\pi_{\lambda+\rho}$ *be the same as in Theorem* 1. *Then*

$$[\pi_{\lambda+\rho}|K;\delta]=0 \qquad \textit{for all } \delta=\delta_\eta(\eta\in\mathcal{E}_\lambda).$$

Proof. Let ν be a weight which occurs in $\pi_{\lambda+\rho}|K$ with respect to $\mathfrak{b}_c$. Then $(\nu+\rho_t,\nu+\rho_t)<(\eta+\rho_t,\eta+\rho_t)$ for all $\eta\in\mathcal{E}_\lambda$ where $\rho_t=\frac{1}{2}\sum_{\alpha\in P_t}\alpha$.

So we have Lemma 1.

Lemma 2. *Let* $\delta=\delta_\kappa$ *be the irreducible representation of* K *with the highest weight* $\kappa=\kappa_1\lambda_1+\cdots\kappa_l\lambda_l$ *on* $\mathfrak{b}$.

For the restriction $\delta|M$ *of representation* δ *of* K *to* M,

$$\delta_\kappa|M=\bigoplus_{\kappa_1\geqq\nu_2\geqq\cdots\geqq\nu_l\geqq|\kappa_l|}\pi'_\nu$$

where π'_ν is the irreducible representation of M with highest the weight $\nu=\nu_2\lambda_2+\cdots+\nu_l\lambda_l$.

For the proof of Lemma 2, see [1].

Proof of Theorem 2. By Lemma 1, $(\text{trace } \pi_{\lambda+\rho})(f^\delta)=0$ for all $\delta=\delta_\eta(\eta\in\mathcal{E}_\lambda)$. By Lemma 2 and Frobenius reciprocity theorem applied to the induced representation $V_{s(\lambda+\rho)}|K$, we have

$$[V_{s(\lambda+\rho)}|K;\delta]=0 \qquad \text{if } s_0\neq s\in W_1,$$

and

$$[V_{s_0(\lambda+\rho)}|K;\delta]=1 \qquad \text{for any } \delta=\delta_\eta(\eta\in\mathcal{E}_\lambda).$$

So we have Theorem 2.

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