2. *Eigenfunction Expansions for Symmetric Systems of First Order in the Half-Space* R_+^n

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1. Introduction. Eigenfunction expansion theory by distorted plane waves was initiated by T. Ikebe [1] and has been investigated by many authors, for example, Y. Shizuta [9], N. A. Shenk II [8], K. Mochizuki [6], J. R. Schulenberger and C. H. Wilcox [7] and others. T. Ikebe treated the Schrödinger operator $-\Delta+q(x)$ in the whole 3-dimensional Euclidean space R^3. Y. Shizuta treated $-\Delta$ in an exterior domain of R^3 and N. A. Shenk II generalized the result to the higher dimensional case (see also T. Ikebe [2]). K. Mochizuki treated symmetric systems in an exterior domain of R^n and J. R. Schulenberger and C. H. Wilcox in the whole space R^n. An other approach to spectral representations for the operators associated with the wave equation and symmetric hyperbolic systems in an exterior domain of R^n is developed by P. D. Lax and R. S. Phillips [3]. In this note we consider stationary problems for symmetric hyperbolic systems with constant coefficients in the half-space R_+^n and give an expansion theorem by the improper eigenfunctions for such a problem. We note that this problem cannot be regarded as a perturbation of the whole space problem. In fact, our theory is a generalization of the sine and cosine transformations in the L^2 space on the positive half-line which are eigenfunction expansions for $-\dfrac{d^2}{dx^2}$ with Dirichlet or Neumann conditions.

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2. Assumptions. We denote the n-dimensional Euclidean space by R^n and its point by $x=(x_1, \cdots, x_n)$. We also denote a point in R^{n-1} by $x'=(x_1, \cdots, x_{n-1})$ and the set $\{x \in R^n ; x_n>0\}$ by R_+^n. Let L be a first order symmetric hyperbolic operator with constant coefficients:

$$L=I\frac{\partial}{\partial t}-\sum_{j=1}^{n} A_j\frac{\partial}{\partial x_j}, \tag{1}$$

where I is the identity matrix of order N and the A_j are $N\times N$ constant Hermitian matrices. We consider the mixed initial and boundary value problem in R_+^n for L:

$$\text{(2)}\qquad \begin{aligned} &L[u(t,x)]=f(t,x), && t>0,\ x\in \boldsymbol{R}^n_+,\\ &u(0,x)=u_0(x), && x\in \boldsymbol{R}^n_+,\\ &Bu(t,x)|_{x_n=0}=0, && t>0, \end{aligned}$$

where $u(t, x)$, $f(t, x)$ and $u_0(x)$ are vector-valued functions whose values lie in the N-dimensional complex space $\boldsymbol{C}^N$ and B is an $l\times N$ constant matrix with rank l. Replacing $u(t, x)$ and $f(t, x)$ in (2) by $e^{i\sigma t}v(x)$ and $-ie^{i\sigma t}g(x)$, respectively, we obtain the corresponding stationary problem:

$$\text{(3)}\qquad \begin{aligned} &(A-\sigma I)v(x)=g(x), \qquad x\in \boldsymbol{R}^n_+,\\ &Bv(x)|_{x_n=0}=0, \end{aligned}$$

where

$$\text{(4)}\qquad A=\frac{1}{i}\sum_{j=1}^{n} A_j\frac{\partial}{\partial x_j}.$$

Put

$$\text{(5)}\qquad A(\eta)=\sum_{j=1}^{n}\eta_j A_j,$$

$$\text{(6)}\qquad M(\xi\,;\lambda)=A_n^{-1}\Big(\lambda I-\sum_{j=1}^{n-1}\xi_j A_j\Big),$$

where we assume that A_n is non-singular and η_j, ξ_k $(1\le j\le n, 1\le k\le n-1)$ are real parameters and λ is a complex parameter. Put

$$\text{(7)}\qquad P(\lambda,\eta)=\det(\lambda I-A(\eta))=\det(-A_n)\det(\tau I-M(\xi\,;\lambda)),$$

where $\eta=(\eta_1,\cdots,\eta_n)=(\xi_1,\cdots,\xi_{n-1},\tau)$. Let

$$\text{(8)}\qquad P(\lambda,\eta)=Q_1(\lambda,\eta)^{m_1}\cdots Q_q(\lambda,\eta)^{m_q}$$

be the factorization of the homogeneous polynomial $P(\lambda,\eta)$ in (λ,η) into powers of distinct factors $Q_j(\lambda,\eta)$ which are irreducible over $\boldsymbol{C}$. $P(\lambda,\eta)$ is monic as a polynomial in λ. Thus $Q_j(\lambda,\eta)$ may be uniquely defined, apart from their order, by requiring that they also be monic in λ and $Q_j(\lambda,\eta)$ are homogeneous. Put

$$\text{(9)}\qquad Q(\lambda,\eta)=Q_1(\lambda,\eta)\cdots Q_q(\lambda,\eta).$$

Definition 1. The operator L is called uniformly propagative if the roots $\lambda_j(\eta)$, $1\le j\le\mu$, of $Q(\lambda,\eta)=0$ satisfy the following conditions:

(i) $\lambda_j(\eta)$, $1\le j\le\mu$, are distinct for $|\eta|=1$.

(ii) $\lambda_j(\eta)=0$ for any $\eta\in\boldsymbol{R}^n$ if there exists some η with $|\eta|=1$ such that $\lambda_j(\eta)=0$.

Here μ denotes the degree of $Q(\lambda,\eta)$ with respect to λ and $|\cdot|$ the Euclidean norm (see, [4] and [10]).

Let $E^+(\xi\,;\lambda)$ be the subspace of $\boldsymbol{C}^N$ spanned by the root vectors corresponding to the characteristic roots of the matrix $M(\xi\,;\lambda)$ with positive imaginary part.

Now we state precisely the assumptions that we impose on L and B:

(L.1) The operator L is uniformly propagative.

(L.2) The operator A is elliptic, i.e., $P(0,\eta)\neq 0$ for any $\eta\in \boldsymbol{R}^n$ with $|\eta|=1$.

(L.3) For any real $\lambda\neq 0$ and any $\xi\in \boldsymbol{R}^{n-1}$ the real roots of $Q(\lambda,\xi,\tau)=0$ with respect to τ are at most double and the number of the real double roots for arbitrarily fixed $(\lambda,\xi)\neq(0,0)$ is at most one.

(B.1) The boundary matrix B is minimally conservative, i.e., $A_n\zeta\cdot\bar{\zeta}=0$ for any $\zeta\in\mathscr{B}\equiv\ker B$[1] $\subset \boldsymbol{C}^N$ and if $\mathscr{E}$ is a subspace of $\boldsymbol{C}^N$ such that $\mathscr{E}\supset\mathscr{B}$ and $A_n\zeta\cdot\bar{\zeta}=0$ for any $\zeta\in\mathscr{E}$, $\mathscr{B}=\mathscr{E}$ holds. Here $x\cdot y$, $x,y\in\boldsymbol{C}^N$, denotes the real inner product.

(B.2) $E^+(\xi:k)\cap\mathscr{B}=\{0\}$ holds for any $\xi\in\boldsymbol{R}^{n-1}$ and any real k with $|\xi|+|k|\neq 0$.

Remark 2. The conditions (L.1) and (L.2) imply that the distinct characteristic roots $\lambda_j(\eta)$, $1\leq j\leq\mu$, of the matrix $A(\eta)$ have constant multiplicities and that μ is even. Thus we put $\mu=2\rho$ and can label $\{\lambda_j(\eta)\}$ in decreasing order:

$$\begin{aligned}&\lambda_1(\eta)>\lambda_2(\eta)>\cdots>\lambda_\rho(\eta)>0>\lambda_{\rho+1}(\eta)>\cdots>\lambda_{2\rho}(\eta),\\ &\lambda_{j+\rho}(\eta)=-\lambda_{\rho-j+1}(-\eta),\qquad 1\leq j\leq\rho.\end{aligned}\tag{10}$$

Moreover we see that N is even. Thus we put $N=2m$. The condition (B.1) implies that $l=m$.

Remark 3. It follows from the condition (B.1) that the operator A defined on $D\equiv\{f\in\mathscr{E}^1_{L^2}(\boldsymbol{R}^n_+)\,;\,Bf|_{x_n=+0}=0\}$ by (4) is essentially self-adjoint in $\boldsymbol{L}^2(\boldsymbol{R}^n_+)\equiv(L^2(\boldsymbol{R}^n_+))^{2m}$. Here $\mathscr{E}^s_{L^2}(\boldsymbol{R}^n_+)$ denotes the space of vector-valued functions whose derivatives of order up to and including s belong to $\boldsymbol{L}^2(\boldsymbol{R}^n_+)$. We denote the self-adjoint extension of A by the same latter.

Remark 4. The condition (L.2) implies that the matrices A_j, $1\leq j\leq n$, are non-singular.

3. Eigenfunctions. Let $G(x,y\,;\lambda)$ be the Green function for $(A-\lambda)$, $\operatorname{Im}\lambda\neq 0$, constructed in [5]. We define projections $P_j(\eta)$, $1\leq j\leq 2\rho$, by

$$P_j(\eta)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda-\lambda_j(\eta)|=\delta}(\lambda I-A(\eta))^{-1}d\lambda, & \eta\neq 0,\\ 0, & \eta=0,\end{cases}\tag{11}$$

where δ is chosen sufficiently small such that the set $\{\lambda\,;\,|\lambda-\lambda_j(\eta)|\leq\delta\}$ contains no roots of $Q(\lambda,\eta)=0$ except $\lambda_j(\eta)$.

Definition 5. Let $\operatorname{Im}\lambda\neq 0$, $x\in\boldsymbol{R}^n_+$ and $\eta\in\boldsymbol{R}^n$. Define

$$\Psi_j(x,\eta\,;\lambda)=\overline{\mathscr{F}}_y[G(x,y\,;\lambda)](\eta)(\lambda_j(\eta)-\lambda)P_j(\eta),\tag{12}$$

$$\Psi_j^{\pm}(x,\eta)=\Psi_j(x,\eta\,;\lambda_j(\eta)\pm i0),\qquad 1\leq j\leq 2\rho.\tag{13}$$

Here we define $G(x,y\,;\lambda)=0$ for $x\in\boldsymbol{R}^n_+$ and $y\notin\boldsymbol{R}^n_+$ and $\overline{\mathscr{F}}_y[f(y)](\eta)$ denotes the conjugate Fourier transform of $f(y)$ in $\mathscr{S}'$ which consists of the temperate distributions.

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1) $\ker B$ denotes the subspace $\{\zeta\in\boldsymbol{C}^N\,;\,B\zeta=0\}$.

It follows from a representation formula of $G(x, y; \lambda)$ that $\Psi_j^{\pm}(x, \eta)$ are well-defined for every $x \in \boldsymbol{R}_+^n$ and almost every $\eta \in \boldsymbol{R}^n$. $\Psi_j^{\pm}(x, \eta)$ are (improper) eigenfunctions for the operator A, i.e., they satisfy $A_x\Psi_j^{\pm}(x, \eta)=\lambda_j(\eta)\Psi_j^{\pm}(x, \eta)$ and $B\Psi_j^{\pm}(x, \eta)|_{x_n=0}=0$.

4. Expansion theorem.

Theorem 6. *Let the conditions* (L.1)$\sim$(L.3) *and* (B.1), (B.2) *be satisfied.*

(i) *For all $f \in \boldsymbol{L}^2(\boldsymbol{R}_+^n)$ the following expansion formula holds:*

$$f(x)=\sum_{j=1}^{2\rho}\int_{\boldsymbol{R}^n}\Psi_j^{\pm}(x, \eta)\hat{f}_j^{\pm}(\eta)d\eta, \tag{14}$$

$$\hat{f}_j^{\pm}(\eta)=\int_{\boldsymbol{R}_+^n}\Psi_j^{\pm}(x, \eta)^* f(x)dx. \tag{15}$$

Here the integrals (14) *and* (15) *are taken in the sense of limit in the mean.*

(ii) *$f \in D(A)$*[2] *if and only if $\hat{f}_j^{\pm}(\eta)$, $\lambda_j(\eta)\hat{f}_j^{\pm}(\eta) \in P_j(\eta)\boldsymbol{L}^2(\boldsymbol{R}^n) \equiv \{f(\eta) \in \boldsymbol{L}^2(\boldsymbol{R}^n)\,;\, P_j(\eta)f(\eta)=f(\eta)\}$, $1\leq j\leq 2\rho$. Then we have*

$$(Af)(x)=\sum_{j=1}^{2\rho}\int_{\boldsymbol{R}^n}\lambda_j(\eta)\Psi_j^{\pm}(x, \eta)\hat{f}_j^{\pm}(\eta)d\eta, \tag{16}$$

$$(Af)_j^{\wedge\pm}(\eta)=\lambda_j(\eta)\hat{f}_j^{\pm}(\eta). \tag{17}$$

From our proof of the above expansion theorem we can see that $\sigma(A)=\sigma_{ac}(A)=\boldsymbol{R}^1$, where $\sigma(A)$ and $\sigma_{ac}(A)$ denote the spectrum and the absolutely continuous spectrum of A, respectively. Moreover we can obtain the explicit representations of the eigenfunctions $\Psi_j^{\pm}(x, \eta)$. Let $\Phi_j : \boldsymbol{L}^2(\boldsymbol{R}_+^n)\to \boldsymbol{L}^2(\boldsymbol{R}^n)$ be the mappings defined by

$$\Phi_j^{\pm}f=\hat{f}_j^{\pm} \quad \text{for all } f \in \boldsymbol{L}^2(\boldsymbol{R}_+^n), \quad 1\leq j\leq 2\rho. \tag{18}$$

Put

$$\Phi\equiv\sum_{j=1}^{2\rho}\Phi_j^{\pm}. \tag{19}$$

Then we can prove that $\Phi_j^{\pm}$ and $\Phi^{\pm}$ are isometries and give explicitly the ranges of $\Phi_j^{\pm}$ and $\Phi^{\pm}$.

5. Outline of proof. The self-adjoint operator A admits a uniquely determined spectral resolution:

$$A=\int_{-\infty}^{\infty}\lambda dE(\lambda), \tag{20}$$

where $\{E(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the right-continuous spectral family of A. Then it follows from the Stieltjes inversion formula that for $f \in \mathscr{D}(\boldsymbol{R}_+^n)$[3] and $a<b$

$$\begin{aligned}\left(\left\{\frac{E(b)+E(b-0)}{2}-\frac{E(a)+E(a-0)}{2}\right\}f, f\right)_+ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\sum_{j=1}^{2\rho}\int_{\boldsymbol{R}^n}d\eta \\ &\times\int_a^b dk\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}\,|\hat{f}_j(\eta\,;\,k\pm i\varepsilon)|^2,\end{aligned} \tag{21}$$

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2) $D(A)$ denotes the domain of the operator A.

3) $\mathscr{D}(\Omega)$ denotes the space of infinitely differentiable vector-valued functions defined on Ω whose supports are compact subsets of Ω.

where $(\ ,\ )_+$ denotes the inner product of $L^2(\boldsymbol{R}^n_+)$ and

$$\hat{f}_j(\eta\,;\lambda)=\int_{\boldsymbol{R}^n_+}\Psi_j(x,\eta\,;\lambda)^*f(x)dx,\quad \mathrm{Im}\,\lambda\neq 0,\quad 1\leq j\leq 2\rho. \tag{22}$$

In order to prove the expansion theorem it suffices to show that we can interchange the order of $\lim_{\varepsilon\downarrow 0}$ and $\int_{\boldsymbol{R}^n}d\eta$ in (21). On the other hand we have that

$$\begin{aligned}\Psi_j(x,\eta\,;\lambda)&=(2\pi)^{-n/2}e^{ix\cdot\eta}P_j(\eta)\\ &\quad-\frac{1}{i}(2\pi)^{-1/2}\mathcal{F}_{y'}[G(x,y'+0\,;\lambda)](\xi)A_nP_j(\eta).\end{aligned} \tag{23}$$

Thus, the part most involved of our study is to analyse the behavior around the singular points of the second term on the right hand side of (23). But its description will be long enough, so that in a forthcoming paper we shall give the more detailed exposition of the content of this note and the proofs. We shall also give in it further results in the case where both the conditions (L.2) and (B.2) are not assumed. When the condition (B.2) is removed, new eigenfunctions corresponding to boundary waves may arise in general.

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