# 33. *On the Injective Radius of Noncompact Riemannian Manifolds*

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In this note all Riemannian manifolds which we deal are connected and complete. Let $M$ be a Riemannian manifold and $p \in M$. $C(p)$ (respectively $Q(p)$) denotes the cut locus (respectively the first conjugate locus) of $p$ in $M$. For $p, q \in M$, $d(p, q)$ denotes the metric distance between $p$ and $q$. As is well known, the function $i: M \to R \cup \{\infty\}$ defined by $i(p) := \min_{q \in C(p)} d(p, q)$ is continuous and $i(p)$ is called the injective radius of $\exp_p$ where $\exp_p: T_p(M) \to M$ is the exponential mapping. If $M$ is compact, then under some conditions, several estimations of the injective radius are obtained, see [3]. Recently in [4], Toponogov asserted that if $M$ is a noncompact Riemannian manifold and for all tangent two plane $\sigma$ its sectional curvature $K_\sigma$ satisfy the inequality $0 < K_\sigma \leqq \lambda$ then for all $p \in M$

$$(1) \qquad i(p) \geqq \frac{\pi}{\sqrt{\lambda}}.$$

Furthermore he asserted that if $M$ is noncompact and $0 \leqq K_\sigma \leqq \lambda$ for all $\sigma$, then there exists a positive $L$ such that for all $p \in M$

$$(2) \qquad i(p) \geqq L.$$

In this note, we give an another proof of estimation (1) by the result of Cheeger and Gromoll [2] and we show that this method remains valid for some two dimensional noncompact Riemannian manifolds.

Every geodesic is always parametrized with respect to arclength. A geodesic $c: [0, \infty) \to M$ is called a ray, if any segment of $c$ is minimal. A subset $A$ of $M$ is called totally convex, if for any $p, q \in A$, any geodesic segment joining $p$ and $q$ is contained in $A$. Let $A$ be a non-empty closed totally convex subset of $M$. Then $A$ is an imbedded topological submanifold of $M$ with totally geodesic interior and possibly nonsmooth boundary $\partial A$, which might be empty, see [2]. Let $M$ be a noncompact manifold of nonnegative sectional curvature. Then the following facts were also proved in [2]. Let $C$ be a closed totally convex subset of $M$. If $\partial C \neq \emptyset$, we set

$$C^a := \{p \in C : d(p, \partial C) \geqq a\}$$
$$C^{\max} := \bigcap_{C^a \neq \emptyset} C^a.$$

Then for any $a \geqq 0$, $C^a$ is totally convex and $\dim C^{\max} < \dim C$. For any

$p \in M$, there exists a family of compact totally convex sets $C_t$, $t \geqq 0$ such that

1) $t_2 \geqq t_1$ implies $C_{t_2} \supset C_{t_1}$ and

$$C_{t_1} = \{q \in C_{t_2} : d(q, \partial C_{t_2}) \geqq t_2 - t_1\}$$

in particular, $\partial C_{t_1} = \{q \in C_{t_2} : d(q, \partial C_{t_2}) = t_2 - t_1\}$,

2) $\bigcup_{t \geqq 0} C_t = M$,

3) $p \in C_0$ and if $\partial C_0 \neq \emptyset$, then $p \in \partial C_0$.

We set $C(0) := C_0$ and if $\partial C(0) \neq \emptyset$, we set $C(1) := C(0)^{\max}$. Inductively we set $C(i+1) := C(i)^{\max}$, if $\partial C(i) \neq \emptyset$. Then there exists integer $k \geqq 0$ such that $\partial C(k) = \emptyset$. $C(k)$ will be called a soul of $M$ and denoted by $S$. If $M$ is homeomorphic to $n$-dimensional Euclidean space $E^n$, then any soul of $M$ is one point set, see [2].

**Theorem 1.** *Let $M$ be a noncompact $n$-dimensional Riemannian manifold.*

1) *if $0 < K_\sigma \leqq \lambda$ for all tangent plane $\sigma$, then*

$$i(p) \geqq \frac{\pi}{\sqrt{\lambda}} \qquad \text{for all } p \in M,$$

2) *if $M$ is homeomorphic to $E^2$ and $0 \leqq K \leqq \lambda$ where $K$ is the Gaussian curvature of $M$, then $i(p) \geqq \pi/\sqrt{\lambda}$ for all $p \in M$.*

**Remark.** If $M$ is noncompact and $0 < K_\sigma$, then following [2], $M$ is diffeomorphic to $E^n$. So 2) can be considered as a generalization of 1) for the case $n=2$. We do not know whether 2) is true for all $n \geqq 2$.

**Proof.** For the present we may assume that $M$ is diffeomorphic to $E^n$ and have the sectional curvature $0 \leqq K_\sigma \leqq \lambda$. We assume that there exist a point $q_0 \in M$ such that $i(q_0) < \pi/\sqrt{\lambda}$. As is mentioned above, for $q_0 \in M$, there exists a family of compact totally convex sets $\{C_t\}_{t \geqq 0}$ such that $q_0 \in C_0$. Let $S = \{s\}$ be a soul of $M$ obtained from $\{C_t\}_{t \geqq 0}$. $C_0$ is compact, so there exists a point $q_1 \in C_0$ such that

$$i(q_1) = \min\{i(q) : q \in C_0\}.$$

Then $i(q_1) \leqq i(q_0) < \pi/\sqrt{\lambda}$. By the assumption, sectional curvature satisfies $0 \leqq K_\sigma \leqq \lambda$. So by the Theorem of Morse-Schoenberg and Lemma 2 in [3; p. 226] there exists a geodesic loop $\gamma_1 : [0, 2i(q_1)] \to M$ such that $\gamma_1(0) = \gamma_1(2i(q_1)) = q_1$. Since $C_0$ is totally convex we have $\gamma_1([0, 2i(q_1)]) \subset C_0$. We show $\gamma_1$ is a closed geodesic. For, if $\dot\gamma_1(0) \neq \dot\gamma_1(2i(q_1))$, then by Lemma 2 [3; p. 226], $i(\gamma_1(i(q_1))) < i(q_1)$. This contradicts the choice of $q_1$. Thus $\gamma_1 : [0, 2i(q_1)] \to M$ extends the closed geodesic $\gamma_1 : (-\infty, \infty) \to M$ having the period $2i(q_1)$. We take $t > 0$. Then by [2], the function $\psi : (-\infty, \infty) \to R$ defined by

$$\psi(s) := d(\gamma_1(s), \partial C_t)$$

is concave. So $\psi(s) \equiv l > 0$ for all $s \in (-\infty, \infty)$, because $\psi$ is bounded. Let $c : [0, l] \to M$ be a minimal geodesic from $\gamma_1(0)$ to $\partial C_t$, and $X$ be the parallel field along $c$ such that $X(0) = \dot\gamma_1(0)$. Then by the Comparison

Theorem of Berger, it follows that there exists $\delta>0$, such that for $0\leqq s<\delta$, the curve $c_s(u):=\exp_{c(u)} sX(u)$ has length $\leqq l$ with equality holding for some $s'>0$ if and only if $V:[0,l]\times[0,s']\to M$ defines a flat totally geodesic rectangle where $V(u,s):=c_s(u)$. For each $s$, the length of the curve $c_s$ is not longer than $l$. So $c_s(l)\subset C_t$ for all $s, 0\leqq s<\delta$. On the other hand, by means of a property of convex sets, $c_s(l)\notin \operatorname{int} C_t$. This shows $c_s(l)\in\partial C_t$ and hence length of the curve $c_s$ is equal to $l$. That is, for all $s, 0<s<\delta$, $V([0,1]\times[0,s])$ is a flat totally geodesic submanifold of $M$. Now, we assume that $M$ is noncompact and $0<K_\sigma\leqq\lambda$, then $M$ is diffeomorphic to $E^n$. Then above fact proves 1). We show 2) by contradiction. Let $s\in M$ be a soul of $M$, then $i(s)\geqq\pi/\sqrt{\lambda}$. If $i(s)<\pi/\sqrt{\lambda}$, then by the argument in 1), there exists a geodesic loop $\gamma:[0,2i(s)]\to M$ such that $\gamma(0)=\gamma(2i(s))=s$. Since $\{s\}$ is totally convex, $\gamma([0,2i(s)])\subset\{s\}$. This is a contradiction. We assume that there exists a point $q_0\in M$ such that $i(q_0)<\pi/\sqrt{\lambda}$. Let $\{C_t\}_{t\geqq 0}$ be a family of totally convex set such that $q_0\in C_0$. We set $A:=\{q\in C_0: i(q)=\min_{r\in C_0}\{i(r)\}\}$. Since $A$ is compact, there exists a points $q_1\in A$ such that $d(q_1,\partial C_0)=\max\{d(q,\partial C_0): q\in A\}$. We set $t_1:=d(q_1,\partial C_0)$. Then $i(q_1)\leqq i(q_0)<\pi/\sqrt{\lambda}$. So there exists a closed geodesic $\gamma_1:(-\infty,\infty)\to M$ such that $\gamma_1(0)=\gamma_1(2i(q_1))=q_1$. Since $M$ is homeomorphic to $E^2$, $\partial C_0$ is homeomorphic to a circle. Hence, by the argument in 1) $\gamma_1((-\infty,\infty))=\partial C_0^{t_1}$ where $C_0^{t_1}:=\{q\in C_0: d(q,\partial C_0)\geqq t_1\}$. Let $s_0$ be the soul of $M$ obtained from $C_0$. Then as is mentioned above, $i(s_0)\geqq\pi/\sqrt{\lambda}$. From this fact and by the choice of $q_1$, we can find a point $q_2'\in\operatorname{int} C_0^{t_1}$ such that $\pi/\sqrt{\lambda}>i(q_2')>i(q_1)$. We set $t_2:=d(q_2',\partial C_0^{t_1})$. Let $q_2\in C_0^{t_1+t_2}$ be a point such that $i(q_2)=\min\{i(q); q\in C_0^{t_1+t_2}\}$. Clearly $\pi/\sqrt{\lambda}>i(q_2')\geqq i(q_2)>i(q_1)$. Then as in 1), there exists a closed geodesic $\gamma_2:(-\infty,\infty)\to M$ such that $\gamma_2(0)=\gamma_2(2i(q_2))=q_2$. By the same reason for $\gamma_1$, $\gamma_2((-\infty,\infty))=\partial C_0^{t_1+t_2}$. Hence, by the Theorem of Gauss-Bonnet, we get

$$\iint_{C_0^{t_1}} K dv=\iint_{C_0^{t_1+t_2}} K dv=2\pi,$$

where $K$ is the Gaussian curvature of $M$ and $dv$ is the area element of $M$. This equation means $K\equiv 0$ on $C_0^{t_1}-C_0^{t_1+t_2}$. So, $L(\gamma_1)=L(\gamma_2)$. This contradicts the fact that $L(\gamma_1)<L(\gamma_2)$. Q.E.D.

W. Klingenberg showed the following theorem, see [3; p. 227].

**Theorem** (W. Klingenberg). *Let $M$ be a compact simply connected even dimensional Riemannian manifold and $0<K_\sigma\leqq\lambda$ for all $\sigma$. Then for all $p\in M$,*

$$i(p)\geqq\frac{\pi}{\sqrt{\lambda}}.$$

Let $M$ be a 2-dimensional compact simply connected Riemannian manifold having the Gaussian curvature $0\leqq K\leqq\lambda$. Then $M$ is homeomorphic to sphere $S^2$. By the Comparison Theorem of Berger, just as the proof of the Theorem of Klingenberg, we can easily see $i(p) \geqq \pi/\sqrt{\lambda}$ for all $p \in M$. Summarizing above we get

**Corollary.** *Let $M$ be a simply connected 2-dimensional Riemannian manifold and its Gaussian curvature satisfies $0 \leqq K \leqq \lambda$. Then $i(p) \geqq \pi/\sqrt{\lambda}$ for all $p \in M$.*

We give an application of Theorem 1. Let $M$ be a compact manifold. Then its volume (we denote by Vol $(M)$) is finite. Conversely if $M$ have finite volume, then is $M$ compact? This is not true in general.

**Theorem 2.** *Let $M$ be an $n$-dimensional Riemannian manifold and whose sectional curvature satisfies $0 < K_\sigma \leqq \lambda$ for all $\sigma$ or $M$ be a 2-dimensional Riemannian manifold whose Gaussian curvature satisfies $0 \leqq K \leqq \lambda$. Then $M$ is compact if and only if* Vol $(M)$ *is finite.*

**Proof.** It sufficies to show that if $M$ is noncompact and $0 < K \leqq \lambda$ (or $M$ is a 2-dimensional noncompact Riemannian manifold and $0 \leqq K \leqq \lambda$) then Vol $(M)$ is infinite. Let $p \in M$. Then, since $M$ is noncompact, there exist a ray $c: [0, \infty) \to M$ such that $c(0) = p$. Let $M$ be an $n$-dimensional Riemannian manifold with $0 < K_\sigma \leqq \lambda$, then $i(p) \geqq \pi/\sqrt{\lambda}$ for all $p \in M$. Let $B_r(p)$ denotes the closed metric ball in $M$ around $p$ with radius $r$. $S^n(1/\sqrt{\lambda})$ denotes the $n$-dimensional sphere in $E^{n+1}$ of constant sectional curvature $\lambda$. Then by [1], $\mathrm{Vol}\,(B_{\pi/\sqrt{\lambda}}(p)) \geqq \mathrm{Vol}\,(S^n(1/\sqrt{\lambda}))$. We consider a family of closed balls $\{B_{\pi/\sqrt{\lambda}}(c((2j+1)\pi/\sqrt{\lambda}))\,;\, j=0, 1, 2, \cdots\}$. If $j \neq k$, then $B_{\pi/\sqrt{\lambda}}(c((2j+1)\pi/\sqrt{\lambda})) \cap B_{\pi/\sqrt{\lambda}}(c((2k+1)\pi/\sqrt{\lambda})) = \emptyset$, because $c$ is a ray. Hence $\mathrm{Vol}\,(M) \geqq \sum_{j=0}^{\infty} \mathrm{Vol}\,(B_{\pi/\sqrt{\lambda}}(c(2j+1)\pi/\sqrt{\lambda})) \geqq \mathrm{Lim}_{j\to\infty}\, j \cdot \mathrm{Vol}\,(S^n(1/\sqrt{\lambda})) = \infty$.

If $M$ is 2-dimensional and $0 \leqq K \leqq \lambda$, then by Classification Theorem in [1], $M$ is isometric to a cylinder or a flat open möbius band or $P^2$ which is homeomorphic to $E^2$. If $M$ is homeomorphic to $E^2$, then by Theorem 1, $i(p) \geqq \pi/\sqrt{\lambda}$ for all $p \in M$. And just as in above we see $\mathrm{Vol}\,(M) = \infty$. Q.E.D.

**Remark.** If Toponogov's result in [4] is true, then Theorem 2 is true for all manifolds satisfying $0 \leqq K_\sigma \leqq \lambda$.

## References


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