29. *Remarks on Moduli of Invertible Elements in a Function Algebra*

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Let X be a compact Hausdorff space and let A be a function algebra on X. A theorem of Hoffman-Wermer [4] asserts as follows: the set of real parts $\operatorname{Re} A$ of A closed implies $A=C(X)$. On the other hand, E. Gorin [3] proved that if A is a function algebra on a compact metric space and if $\log|A^{-1}|=C_R(X)$, then $A=C(X)$, where A^{-1} denotes the set of invertible elements in A and $|A^{-1}|$ is the moduli of A^{-1}. We here consider the following condition: there is a closed subset N in $C_R(X)$ with $\log|A^{-1}|\supset N\supset \operatorname{Re} A$. The hypotheses of two above theorems satisfy the condition. The aim of this paper is to investigate properties of function algebras which have the condition and to give an extension of the Gorin theorem (Corollary 2).

We begin with the following theorem which states relations between the Hoffman-Wermer theorem and the Gorin theorem.

Theorem. *Let X be a compact Hausdorff space and let A be a function algebra on X. Assume that there is a closed linear subspace in $C_R(X)$ with $\log|A^{-1}|\supset N\supset \operatorname{Re} A$. If F is a maximal antisymmetric set for A, then the following properties are equivalent.*

(i) *F is metric*

(ii) $N|F=\operatorname{Re}(A|F)$

(iii) *F is a single point,*

where $A|F$ and $N|F$ denote the restrictions of A and N the F respectively.

In order to verify the theorem, we shall first show the following lemma which is essentially due to Hoffman-Wermer [4].

Lemma. *Let A be a function algebra on a compact Hausdorff space. Assume that there is a closed linear subspace N in $C_R(X)$ with $N\supset \operatorname{Re} A$. If F is a maximal antisymmetric set for A and if $N|F=\operatorname{Re}(A|F)$, then F is a single point.*

Proof. If $h\in \operatorname{Re}(A|F)$, then from the hypothesis there is an $h^*\in N$ such that $h^*|F=h$. Therefore we have:

$$\varphi\to\varphi|F$$

is a linear transformation of N onto $\operatorname{Re}(A|F)$ whose norm is 1. Let $R=\{f\in N; f(F)=0\}$. Then R is a closed linear subspace and the factor space N/R is isometric to $\operatorname{Re}(A|F)$. This can be verified by the following fact which we can find in Hoffman-Wermer [4]: for any $h \in \mathrm{Re}(A|F)$ and for any $\varepsilon > 0$, there is an $f^* \in A$ such that $(\mathrm{Re} f^*)|F = h$ and $\|\mathrm{Re} f^*\| \leqq \|h\| + \varepsilon$. Therefore $\mathrm{Re}(A|F)$ is closed in $C_R(F)$ because N/R is complete. This shows that F is a single point (cf. [4]).

Proof of Theorem. Since (ii)→(iii) is clear from Lemma, we have only to prove that (i) implies (ii). We denote by A_F the restriction of A to F and regard A_F as a function algebra on F. A_F^{-1} denotes the set of invertible elements in A_F and E_F the set of exponentials of elements of A_F. If (ii) is not satisfied, then since F is metric, it suffices to show that for any real number λ there is a $g_\lambda \in A_F^{-1}$ such that $g_1 \notin E_F$, $|g_\lambda| = |g_1|^\lambda$ and $g_\lambda(x_0) = 1$ for some $x_0 \in F$ (cf. [2], 70–71 or [5], 129–130). Since $\log|A^{-1}| \supset N \supset \mathrm{Re} A$, we have

$$\log|A_F^{-1}| \supset (\log|A^{-1}|)|F \supset N|F \supset \mathrm{Re}(A_F).$$

If we take an f_0 in $N|F - \mathrm{Re}(A_F)$, there is a $g \in A_F^{-1}$ with $f_0 = \log|g|$. It is clear that $g \notin E_F$. We can assume that $g(x_0) = 1$ for some $x_0 \in F$. Now since for any real number λ, $\lambda f_0 = \lambda \log|g| \in N|F \subset \log|A_F^{-1}|$, there is $g_\lambda \in A_F^{-1}$ such that $|g_\lambda| = |g|^\lambda$. From this we obtain desired functions: $\{g_\lambda\} \subset A_F^{-1}$, $g_1 = g$, $g_1 \notin E_F$, and $g_\lambda(x_0) = 1$.

Remark. If we put $N = \mathrm{Re} A$ in Theorem, (ii) is satisfied.

Corollary 1. *Let A be a function algebra on a compact Hausdorff space X. Assume that there is a closed subset N in $C_R(X)$ (not necessarily linear) with* $\log|A^{-1}| \supset N \supset \mathrm{Re} A$. *If $\{F_\alpha\}$ is the family of maximal antisymmetric sets for A which are not metric, then the essential set E for A is equal to the closure $\overline{\bigcup_\alpha F_\alpha}$ of $\bigcup_\alpha F_\alpha$.*

Proof. Since $\log|A^{-1}| \supset N \supset \overline{\mathrm{Re} A}$ and $\overline{\mathrm{Re} A}$ is a closed linear subspace, by Theorem, a maximal antisymmetric set for A is not metric or a single point. Therefore $\overline{\bigcup_\alpha F_\alpha}$ contains E. Conversely $E = X - P^i$, where P denotes the set of maximal antisymmetric sets for A which consist of a single point and P^i is the interior of P (cf. [6]). So for any α $F_\alpha \subset X - P \subset X - P^i = E$ and we have $E = \overline{\bigcup_\alpha F_\alpha}$.

The following corollary is an extension of the Gorin theorem.

Corollary 2. *Let A be a function algebra on a compact metric space X. If there is a closed subset N in $C_R(X)$ (not necessarily linear) with* $\log|A^{-1}| \supset N \supset \mathrm{Re} A$, *then* $A = C(X)$.

Proof. Since X is metric, so is a maximal antisymmetric set for A. It concludes that $A = C(X)$ by Theorem and Bishop [1].

References

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