28. *Examples of Foliations with Non Trivial Exotic Characteristic Classes*

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1. Introduction. In [1], R. Bott has defined the exotic characteristic classes for foliations as follows:

Let $q \geqq 1$ be an integer.

First, a cochain complex (WO_q, d) is defined. Let $\boldsymbol{R}[c_1, \cdots, c_q]$ denote the graded polynomial algebra over $\boldsymbol{R}$ generated by the elements c_i with degree $2i$. Set

$$\boldsymbol{R}_q[c_1, \cdots, c_q] = \boldsymbol{R}[c_1, \cdots, c_q]/\{\phi ; \deg(\phi) > 2q\}.$$

Let $E(h_1, h_3, \cdots, h_r)$ denote the exterior algebra over $\boldsymbol{R}$ generated by the elements h_i with degree $2i-1$, where r is the largest odd integer $\leqq q$. Then, as a graded algebra over $\boldsymbol{R}$

$$WO_q = \boldsymbol{R}_q[c_1, \cdots, c_q] \otimes E(h_1, h_3, \cdots, h_r)$$

and a unique antiderivation of degree 1

$$d : WO_q \to WO_q$$

is defined by requiring

$$d(c_i) = 0, \qquad i = 1, \cdots, q$$
$$d(h_i) = c_i, \qquad i = 1, 3, \cdots, r.$$

Secondly, given a C^∞-smooth codimension q foliation $(N, \mathcal{F})$ on an oriented manifold N without boundary, a homomorphism of cochain complexes

$$\lambda_{(N,\mathcal{F})} : WO_q \to A_C^*(N)$$

is defined, where $A_C^*(N)$ denotes the space of complex smooth forms on N. We used the notation $\lambda_{(N,\mathcal{F})}$ in place of λ_E of Bott [1]. Here the homomorphism $\lambda_{(N,\mathcal{F})}$ depends only on the choices of two connections on the normal bundle of the foliation $(N, \mathcal{F})$ called metric and basic.

In cohomology, $\lambda_{(N,\mathcal{F})}$ induces a homomorphism of graded $\boldsymbol{R}$-algebras

$$\lambda^*_{(N,\mathcal{F})} : H^*(WO_q) \to H^*(N ; \boldsymbol{C})$$

which does not depend on the choices of the above connections.

The elements of $\lambda^*_{(N,\mathcal{F})}(H^*(WO_q))$ are called the exotic characteristic classes for the foliation $(N, \mathcal{F})$.

In this paper, we construct the examples of foliations with non trivial exotic characteristic classes, that is,

Theorem. *For any integer* $q \geqq 1$, *there exists a* C^∞*-smooth* *codimension q foliation $(M,\mathcal{F})$ on a closed $(2q+1)$-manifold such that all the exotic characteristic classes for the foliation which correspond to the canonical generators of $H^{2q+1}(WO_q)$ are non zero in $H^{2q+1}(M;\boldsymbol{C})$.*

Remark. When $q=1$, the generator $[c_1\cdot h_1]$ of $H^3(WO_1)\cong\boldsymbol{R}$ is called the Godbillon-Vey invariant and R. Roussarie constructed an example of foliation with non trivial Godbillon-Vey invariant (see Bott [1]).

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Detailed proof will appear elsewhere.

2. Construction of $(M,\mathcal{F})$. Throughout this paper, integer $q\geqq 1$ is to be fixed and all foliations are to be C^∞-smooth codimension q foliations.

Let $O(q+1,1)=\{X\in GL(q+2;\boldsymbol{R});\ {}^tXBX=B\}$, where

$$B=\begin{pmatrix} I_{q+1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Define subgroups $H\subset K\subset G$ of the Lie group $O(q+1,1)$ as follows: G is the identity component of $O(q+1,1)$,

$$H=\left\{\begin{pmatrix} X & 0 \\ 0 & I_2 \end{pmatrix};\ X\in SO(q)\right\}$$

$$K=\left\{X=(x_{ij})\in G;\ \begin{array}{l}\det\begin{pmatrix} x_{q+1\,q+1} & x_{q+1\,q+2} \\ x_{q+2\,q+1} & x_{q+2\,q+2}\end{pmatrix}=1, \\ \text{and } x_{i\,q+1}+x_{i\,q+2}=0, \text{ for } i=1,\cdots,q\end{array}\right\}.$$

Then G/H is an open orientable $(2q+1)$-manifold and G/K is a q-manifold.

Set $\overline{M}=G/H$, and $\overline{M}$ is foliated into the fibres of the fibre bundle $\overline{M}=G/H\to G/K$. We denote this foliation by $(\overline{M},\overline{\mathcal{F}})$. Clearly, the foliation $(\overline{M},\overline{\mathcal{F}})$ is a G-invariant foliation of codimension q on $\overline{M}$.

By A. Borel [2], G admits a discrete subgroup D such that the quotient space $M=D\backslash\overline{M}$ is a closed orientable $(2q+1)$-manifold. Since the foliation $(\overline{M},\overline{\mathcal{F}})$ is G-invariant, M has a codimension q foliation $(M,\mathcal{F})$ induced naturally from $(\overline{M},\overline{\mathcal{F}})$.

3. Exotic characteristic classes. It is easy to see the following.

Lemma 1. *Each canonical generator of $H^{2q+1}(WO_q)$ is represented by some $\phi\cdot h_j\in WO_q$, where $\phi\in\boldsymbol{R}_q[c_1,\cdots,c_q]$ is a monomial with degree $2(q-j+1)$.*

In this section, all manifolds are to be oriented manifolds without boundary.

Let (N,g) be a Riemannian manifold and ∇ the Riemannian connection on N. For any foliation $(N,\mathcal{F})$, let $\tau(\mathcal{F})$ (resp. $\nu(\mathcal{F})$) denote the subbundle of $\tau(N)$ tangent (resp. normal) to the foliation. Then a metric connection ∇^0 and a basic connection ∇^1 on $\nu(\mathcal{F})$ are defined as follows:

$$\nabla^0_X(Y)=\pi\nabla_X(Y)$$
$$\nabla^1_X(Y)=\pi[X_{\tau(\mathcal{F})}, Y]+\nabla^0_{X_{\nu(\mathcal{F})}}(Y)$$

for any $X\in\mathfrak{X}(N)$, $Y\in\Gamma(\nu(\mathcal{F}))$, where $\pi:\tau(N)\to\nu(\mathcal{F})$ is the natural projection and $X_{\tau(\mathcal{F})}\in\Gamma(\tau(\mathcal{F}))$, $X_{\nu(\mathcal{F})}\in\Gamma(\nu(\mathcal{F}))$ are such that $X=X_{\tau(\mathcal{F})}+X_{\nu(\mathcal{F})}$.

Then the homomorphism of cochain complexes

$$\lambda_{(N,\mathcal{F})}: WO_q\to A^*_C(N)$$

is uniquely determined from the above connections ∇^0 and ∇^1, hence from the foliation $(N,\mathcal{F})$ and the Riemannian metric g. Thus we denote this $\lambda_{(N,\mathcal{F})}(\omega)$ by $\omega((N,\mathcal{F}),g)$ for $\omega\in WO_q$. Then we have the followings.

Lemma 2. *Let G be a Lie group and N be a G-manifold with a G-invariant Riemannian metric g. If a foliation $(N,\mathcal{F})$ is G-invariant, then $\omega((N,\mathcal{F}),g)$ is also G-invariant for any $\omega\in WO_q$.*

Lemma 3. *Let $\bar{N}$ be a covering manifold over N with projection p. If N has a foliation $(N,\mathcal{F})$. Then,*

$$p^*\omega((N,\mathcal{F}),g)=\omega((\bar{N},p^*\mathcal{F}),p^*g)$$

for any $\omega\in WO_q$ and a Riemannian metric g on N, where $(\bar{N},p^*\mathcal{F})$ (resp. p^*g) is the pull back of $(N,\mathcal{F})$ (resp. g).

4. Outline of the proof of theorem. Let $(\bar{M},\bar{\mathcal{F}})$ and G be as in Section 2. Then the following is a key lemma for the calculation of $\lambda_{(\bar{M},\bar{\mathcal{F}})}$.

Lemma 4. *There exist a G-invariant Riemannian metric g on $\bar{M}$ and a 1-form θ on $\bar{M}$ such that the followings hold:*

(1) *At $o=H\in\bar{M}=G/H$,*

$$c_i((\bar{M},\bar{\mathcal{F}}),g)=\alpha_i(\sqrt{-1}/2\pi)^i(d\theta)^i,\ \alpha_i>0,$$
$$h_j((\bar{M},\bar{\mathcal{F}}),g)=\beta_j(\sqrt{-1}/2\pi)^j(d\theta)^{j-1}\wedge\theta,\ \beta_j>0$$

for $i=1,\cdots,q$, $j=1,3,\cdots,r$.

(2) *The $(2q+1)$-form $(d\theta)^q\wedge\theta$ is non zero at $o\in\bar{M}$.*

It follows from Lemma 1 and Lemma 4 that all the differential forms which correspond to the cochains representing the canonical generators of $H^{2q+1}(WO_q)$ are non zero at $o=H\in\bar{M}=G/H$. Then these differential forms are G-invariant and nowhere zero on $\bar{M}$ by Lemma 2. Hence they induce nowhere zero differential forms on M. In view of Lemma 3, the exotic characteristic classes for $(M,\mathcal{F})$ corresponding to the canonical generators of $H^{2q+1}(WO_q)$ are represented by these differential forms. Therefore they are non zero in $H^{2q+1}(M;\boldsymbol{C})$.

References

[1] R. Bott: Lectures on Characteristic Classes and Foliations. Lecture Notes in Math., No. 279, Springer-Verlag (1972).

[2] A. Borel: Compact Cliford-Klein forms of symmetric spaces. Topology, 2, 111–123 (1963).