49. *On Fixed Point Theorem*

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In this paper we shall prove a fixed point theorem by the method of ranked space. The linear operator in the following theorem is not necessarily continuous. Throughout this note, $g, f, x, y, z, \cdots$ will denote points of a ranked space, $U_i, V_i, \cdots$ neighbourhoods at the origin with rank i and $\{U_{r_i}\}, \{V_{r_i}\}, \cdots$ fundamental sequences of neighbourhoods with respect to the origin. Let a linear space E be a ranked space with indicator ω_0, which satisfies the following conditions:

(E, 1)
(1) For any neighbourhood U_i, the origin belongs to U_i.
(2) For any neighbourhood U_i, and for any integer n, there is an m such that $m \geqq n$ and $U_m \subseteqq U_i$.
(3) The space E is the neighbourhood at the origin with rank zero.

Furthermore we define $g+U_i$ as a neighbourhood at point g with rank i. Then the space E is called a pre-linear ranked space. Moreover the space E having the following conditions (E, 2) and (E, 3), is called a linear ranked space.

(E, 2) The following conditions are the modification of the Washihara's conditions [3].

(R, L_1) For any $\{U_{r_i}\}$ and $\{V_{r'_i}\}$, there is a $\{W''_{r_i}\}$ such that
$$U_{r_i} + V_{r'_i} \subseteqq W_{r''_i}.$$

(R, L_2)$''$ For any $\{U_{r_i}\}$ and any $\lambda > 0$, there are a $\{U_{r'_i}\}$, all of whose members belong to $\{U_{r_i}\}$ and a natural number j such that
$$\lambda U_{r_i} \subseteqq U_{r'_i} \quad \text{for all } i \ (i \geqq j).$$

(E, 3) For any neighbourhood U_i and any λ $(0 \leqq \lambda \leqq 1)$, $\lambda U_i \subseteqq U_i$.

Definition 1 (T_1-space). A pre-linear ranked space E is called a T_1-space if for any g, f $(g \neq f, g \in E, f \in E)$ and any fundamental sequence at the origin $\{U_{r_i}\}$ there exists some U_{r_j} belonging to $\{U_{r_i}\}$ such that $g + U_{r_j} \not\ni f$.

Definition 2 (T_2-space). A pre-linear ranked space E is called a T_2-space if for any g, f $(g \neq f, g \in E, f \in E)$ and any fundamental sequence at the origin $\{U_{r_i}\}$ there exist some U_{r_j} and U_{r_k} belonging to $\{U_{r_i}\}$ such that $(g+U_{r_j}) \cap (f+U_{r_k}) = \phi$.

Lemma 1. *Let E be a T_1 pre-linear ranked space, all of whose neighbourhoods at the origin are symmetric ($U=-U$). Then the space E is a T_2-space.*

Definition 3 (R-convergent). Let E be a pre-linear ranked space. A sequence $\{g_i\}$ in E is called to be R-convergent to g_0 in E with respect to $\{U_{r_i}\}$ if there is a fundamental sequence $\{U_{r'_i}\}$, all of whose members belong to $\{U_{r_i}\}$, such that $g_i \in g_0 + U_{r'_i}$ for all i. We say g_0 to be R-limiting point of $\{g_i\}$.

Definition 4 (R-closure, R-closed set). Let E be a pre-linear ranked space. The R-closure set $\bar{S}$ of subset S in E with respect to $\{U_{r_i}\}$ is the set having the following properties: For any g $(g \in \bar{S})$ there are a sequence $\{g_i\}$ $(g_i \in S)$ and a $\{U_{r'_i}\}$, all of whose members belong to $\{U_{r_i}\}$, such that $g_i \in g + U_{r'_i}$ for all i. If $\bar{S} = S$, we say that S is an R-closed set with respect to $\{U_{r_i}\}$.

Definition 5 (R-compact). Let E be a pre-linear ranked space. A subset S in E is called R-compact with respect to $\{U_{r_i}\}$ if the following conditions are satisfied:
(1) For any g $(g \in S)$ and any U_{r_j} $(U_{r_j} \in \{U_{r_i}\})$ there exists some λ $(\lambda > 0)$ such that $g \in \lambda U_{r_j}$.
(2) If for each g $(g \in S)$ there exists some U_g belonging to $\{U_{r_i}\}$ such that $S \subseteq \bigcup_{g \in S} (g + U_g)$, then we have a finite family $\{g_i\}_{i \in I}$ in S such that $S \subseteq \bigcup_{i \in I} (g_i + U_{g_i})$.

Lemma 2. *Let E be a pre-linear ranked space and let K be an R-compact subset with respect to $\{U_{r_i}\}$. Then for any sequence $\{g_i\}$ $(g_i \in K)$, there exist some subsequence $\{g_{n_i}\}$ from $\{g_i\}$ and a point g in K such that $\{g_{n_i}\}$ is R-convergent to g with respect to $\{U_{r_i}\}$. Moreover the set K is an R-closed set with respect to $\{U_{r_i}\}$, if the space E is T_2-space.*

Lemma 3. *Let E and F be two pre-linear ranked spaces and let a subset K be R-compact with respect to $\{U_{r_i}\}$ in E. Suppose T is a linear operator with $D(T) = D \supset K$, having the following property: There exists a fundamental sequence of neighbourhoods at the origin in F, $\{W_{r'_i}\}$ such that $T(U_{r_i} \cap D) \subseteq W_{r'_i}$ for all i. Then $T(K)$ is R-compact with respect to $\{W_{r'_i}\}$.*

Proof. (1) Since K is R-compact, for any $T(g)$ $(g \in K)$ and any $W'_{r'_j}$ $(W_{r'_j} \in \{W'_{r_i}\})$ there exists some λ $(\lambda > 0)$ such that

$$T(g) \in T(\lambda(U_{r_j} \cap D)) \subseteq \lambda W_{r'_j}.$$

(2) Suppose that for each $f (f \in T(K))$ there exists some W_f belonging to $\{W_{r'_i}\}$ such that $T(K) \subseteq \bigcup_{f \in T(k)} (f + W_f)$. By our hypotheses, for W_f and any g $(g \in T^{-1}(f))$ there exists some U_g $(U_g \in \{U_{r_i}\})$ such that $\bigcup_{g \in T^{-1}(f)} (g + U_g \cap D) \subseteq T^{-1}(f + W_f)$. Then we have

$$K \subseteq \bigcup_{f \in T(k)} \left[\bigcup_{g \in T^{-1}(f)} (g + U_g \cap D) \right] \subseteq \bigcup_{f \in T(k)} T^{-1}(f + W_f).$$

Since K is R-compact, this Lemma is true.

Fixed point theorem.

Let E be a T_1 linear ranked space with the following properties:

(1) A neighbourhood at the origin is symmetric.

(2) If $\{V_{r_i}\}$ is any fundamental sequence of neighbourhoods at the origin, for any member U belonging to $\{V_{r_i}\}$ and any element x $(x \in U)$ there is some $V(V \in \{V_{r_i}\})$ such that $x+V \subset U$.

And let non-empty convex subset K be R-compact with respect to $\{U_{r_i}\}$. Suppose T is a linear operator from E to E, whose domain is an invariant subspace D, such that $D \supseteqq K$, $T(K) \subseteqq K$ and if $\{V_{r_i}\}$ is any fundamental sequence of neighbourhoods at the origin, for any integer m $(m>0)$ and any two distinct points x, y $(x, y \in D)$ there is some $U(U \in \{V_{r_i}\})$ such that $x+(1/m)(U \cap D+T(U \cap D)+\cdots+T^m(U \cap D)) \not\ni y$. Then there is some $v_0(v_0 \in K)$, for which $T(v_0)=v_0$.

Proof. For any positive integer n, we define $T_n=(1/n)\sum_{k=0}^{n-1} T^k$, where T^0 is the identity. Since K is convex, $T_n(K) \subseteqq K$. As $T_n(T_m x) = T_m(T_n x)$ for $x \in K$, $K \cap T_2(K) \cap \cdots \cap T_n(K)$ is non-empty. Since K is R-compact with respect to $\{U_{r_i}\}$, for U_{r_1} there is a finite sequence $x_1^{(1)} \cdots x_{i_1}^{(1)}$ contained in K such that $K \subseteqq \bigcup_{j=1 \cdots i_1}(x_j^{(1)}+U_{r_1})$. For brevity, we denote $K \cap (x_j^{(1)}+U_{r_1})=A_j^{(1)}$. Hence each point x in K belongs to an at most finite member of the above-mentioned sets $(x_j^{(1)}+U_{r_1})$. Thus there is a V_x $(V_x \in \{U_{r_i}\})$, whose rank is larger than γ_1, such that $x+V_x \subseteqq (x_j^{(1)}+U_{r_1})$ whenever $x \in (x_j^{(1)}+U_{r_1})$. Next, let V_x be the foregoing neighbourhood corresponding to each point x in K. Then we have $K \subseteqq \bigcup_{x \in k}(x+V_x)$. Since K is R-compact, there is a finite sequence $x_1^{(2)} \cdots x_{i_2}^{(2)}$ contained in K such that $K \subseteqq \bigcup_{j=1 \cdots i_2}(x_j^{(2)}+U_{(2)}^{(j)})$, where $U_{(2)}^{(j)} = V_{x_j^{(2)}}$. For brevity, we denote $K \cap T_2(K) \cap (x_j^{(2)}+U_{(2)}^{(j)})=A_j^{(2)}$. Continue this process, then we have some finite family of sets $\{A_j^{(n)}\}_{j=1 \cdots i_n}$ for any positive integer n, where $A_j^{(n)}=K \cap T_2(K) \cap \cdots \cap T_n(K) \cap (x_j^{(n)}+U_{(n)}^{(j)})$ and $\bigcup_{j=1}^{i_n} A_j^{(n)}=K \cap T_2(K) \cap \cdots \cap T_n(K) \neq \phi$. Thus there is some finite sequence of positive integers $j_1, \cdots, j_n$ such that $A_{j_1}^{(1)} \supseteqq \cdots \supseteqq A_{j_n}^{(n)}$ and $A_{j_n}^{(n)} \neq \phi$. Hence if for $A_j^{(1)}$ we put $I(A_j^{(1)})=\sup\{\alpha : \exists A_{j_i}^{(i)}$ $(i=2, \cdots, \alpha)$ such that $A_j^{(1)} \supseteqq A_{j_2}^{(2)} \supseteqq \cdots \supseteqq A_{j_\alpha}^{(\alpha)}$ & $A_{j_\alpha}^{(\alpha)} \neq \phi\}$, there is some $A_{j_1}^{(1)}$ such that $I(A_{j_1}^{(1)})=\infty$. Next, for any $A_j^{(2)}$ contained in the above $A_{j_1}^{(1)}$ with $I(A_{j_1}^{(1)}) =\infty$, we put $I(A_{j_1}^{(1)}, A_j^{(2)})=\sup\{\alpha : \exists A_{j_i}^{(i)}$ $(i=3, \cdots, \alpha)$ such that $A_{j_1}^{(1)} \supseteqq A_j^{(2)} \supseteqq A_{j_3}^{(3)} \supseteqq \cdots \supseteqq A_{j_\alpha}^{(\alpha)}$ & $A_{j_\alpha}^{(\alpha)} \neq \phi\}$. We call $I(A_{j_1}^{(1)})$ and $I(A_{j_1}^{(1)}, A_j^{(2)})$ the characters of $A_{j_1}^{(1)}$ and $\{A_{j_1}^{(1)}, A_j^{(2)}\}$, respectively. Thus, if we continue the foregoing process, we have a sequence of sets $A_{j_1}^{(1)}, A_{j_2}^{(2)}, \cdots, A_{j_i}^{(i)}, \cdots$ such that the characters of $\{A_{j_1}^{(1)}, \cdots, A_{j_i}^{(i)}\}$ for each i is infinite. This means that there is a sequence of sets $A_{j_1}^{(1)} \supseteqq A_{j_2}^{(2)} \supseteqq \cdots \supseteqq A_{j_i}^{(i)} \supseteqq \cdots$ and $A_{j_i}^{(i)} \neq \phi$ for $i=1, 2, \cdots$. Let $\{v_i\}$ be a sequence of points that $v_i \in A_{j_i}^{(i)}$ for each i. Then, since K is R-compact, there is a subsequence $\{v_{n_i}\}$ from $\{v_i\}$ and a point v_0 in K such that $v_{n_i} \to v_0$ with respect to $\{U_{r_i}\}$. That is, there is a fundamental sequence $\{U_{r_i'}\}$ $(U_{r_i'} \in \{U_{r_i}\})$ such that $v_{n_i} \in v_0+U_{r_i'}$. Thus, for sufficiently large n_j, we have $v_i-v_0=v_i-x_{j_i}^{(i)}+x_{j_i}^{(i)}-v_{n_j}+v_{n_j}$

$-v_0 \in U_{(i)}^{(j_i)} + U_{(i)}^{(j_i)} + U_{r_i'}$. Then there is a fundamental sequence $\{U_{r_i}^*\}$ such that $U_{(i)}^{(j_i)} + U_{(i)}^{(j_i)} + U_{r_i'} \subsetneqq U_{r_i}^*$. Thus we see

$$v_i - v_0 \in U_{r_i}^* \qquad \text{for all } i \tag{1}$$

Now, for any positive integer n, we can make $T_{n+1}(D)$ into a pre-linear ranked space by defining $y + W_i$, where $W_i = T_{n+1}(U_{r_i}^* \cap D)$, as a neighbourhood at y ($y \in T_{n+1}(D)$) with rank i, and $T_{n+1}(D)$ as the neighbourhood with rank zero. It is easily seen that the pre-linear ranked space $T_{n+1}(D)$ is T_2-space. The relation $U_{(i)}^{(j_i)} \subsetneqq U_{r_i}^*$ implies that K is R-compact with respect to $\{U_{r_i}^*\}$. Thus $T_{n+1}(K)$ is R-compact with repect to $\{W_i\}$ by Lemma 3, and is R-closed. As $W_i = T_{n+1}(U_{r_i}^* \cap D) = [1/(n+1)] (U_{r_i}^* \cap D + T(U_{r_i}^* \cap D) + \cdots + T^n(U_{r_i}^* \cap D))$, we see $[1/(n+1)](U_{r_i}^* \cap D) \subsetneqq W_i$. Hence, the relation (1) implies that $[1/(n+1)]v_i - [1/(n+1)]v_0 \in W_i$ for all i. As we have $v_i \in T_{n+1}(K)$ for sufficiently large i, then $[1/(n+1)v_0] \in \overline{[1/(n+1)]T_{n+1}(K)}$, where $\overline{[1/(n+1)]T_{n+1}(K)}$ is R-closure in $T_{n+1}(D)$ with respect to $\{W_i\}$. Since it is easily verified that $\overline{[1/(n+1)]T_{n+1}(K)} = [1/(n+1)]\overline{T_{n+1}(K)}$, we assert $v_0 \in T_{n+1}(K)$ for any n. It can be shown that $K - K$ is a bounded set in E, that is, there is a fundamental sequence $\{V_{r_i}^*\}$ in E and some sequence of numbers $\{\lambda_i\}$ ($\lambda_i > 0$) such that $K - K \subsetneqq \lambda_i V_{r_i}^*$ for all i. Finally, we shall prove that $T(v_0) = v_0$. Suppose it is not true, that is, $T(v_0) \neq v_0$. Since the space E is T_1-space, there is some $V_{r_{i_0}}^*$ in the above sequence $\{V_{r_i}^*\}$ such that $T(v_0) - v_0 \notin V_{r_{i_0}}^*$. However, the fact that $T_n(K) \ni v_0$ for each n implies that there is some z_n ($z_n \in K$) such that $v_0 = T_n(z_n) = (1/n)\sum_{k=0}^{n-1} T^k(z_n)$. Thus we have $T(v_0) - v_0 = (1/n)\sum_{k=1}^{n} T^k(z_n) - (1/n)\sum_{k=0}^{n-1} T^k(z_n) = (1/n)(T^n(z_n) - z_n) \in (1/n)(K-K)$. Hence, we assert $(K-K) \not\subseteq nV_{r_{i_0}}^*$ for all n. This is a contradiction. This completes the proof.

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