67. *Kneser's Property of Solution Families of Non-linear Volterra Integral Equations*

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Consider a system of nonlinear integral equations of Volterra-type

$$\text{(P)} \qquad x(t)=f(t)+\int_0^t g(t,s,x(s))ds.$$

Recently R. K. Miller and G. R. Sell [1] proved some fundamental theorems of (P) under fairly general assumptions on $f(t)$ and $g(t,s,x)$ similar to the Carathèodory-type. They showed that the cross-section

$$F(t)=\{y: y=x(t), \text{ where } x \text{ is some solution of (P)}\}$$

is compact in R^n for all $t\in[0,\alpha_M)$, where α_M is either ∞ or a finite number such that there is a solution $x(t)$ of (P) for which $\limsup_{t\uparrow\alpha_M}|x(t)|=\infty$. This appears to be a generalization of H. Kneser's theorem to integral equations.

For the case where $g(t,s,x)$ is a bounded continuous function of (t,s,x) on $\{0\leqq s\leqq t\leqq\alpha\}\times R^n$, Sato [3] has shown that $F(t)$ is a continuum, i.e., a compact and connected set for all $t\in[0,\alpha]$. One of the present authors later proved in [4] that the family of all solution-curves is a continuum even in $C[0,\alpha]$.

We think that it is interesting to know whether $F(t)$ is a continuum or not for all $t\in[0,\alpha_M)$ under the weaker assumptions of Miller and Sell. The purpose of this note is to give an answer in the affirmative for this problem. Moreover, we can demonstrate that the family of solutions of (P) is also a continuum in the Frèchet space $C[0,\alpha_M)$.[1)]

Since the method we employed in this paper mainly depends on Carathèodory iterates, there is no need in our proof to use the approximate functions g_n to g satisfying the Lipschitz condition which was employed in [3] and [4].

We assume the hypotheses (H1)–(H5) on $f(t)$ and $g(t,s,x)$ used in our previous note [5]. We shall show the following main theorem.

Theorem. *Let the functions f and g satisfy* (H1)–(H5), *then there exists a number $\alpha_M>0$ such that for each $t\in[0,\alpha_M)$ the set $F(t)$ is compact and connected as a subset of R^n. Moreover the number α_M is*

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1) After completing this manuscript, we found that W. G. Kelly (Proc. Amer. Math. Soc., **40**, 1973) proved a local Kneser property, that is, the set $\{x(t)\in C[0,d];\ x(t)$ is a solution of (P) on $[0,d]\}$ is compact and connected in the space $C[0,d]$ for any $d<\alpha_M$.

maximal in the sense that either $\alpha_M=\infty$ or there exists a right maximally defined solution $x(t)$ of (P) *whose domain of definition is the interval* $[0, \alpha_M)$.

Proof. It is sufficient to show that $F(c)$ is connected for each $c \in [0, \alpha_M)$. Suppose the contrary. Since $F(c)$ is compact, $F(c)$ can be expressed as a union of two disjoint nonempty compact sets, i.e., $F(c) = F_1 \cup F_2$, where F_1 and F_2 are nonempty compact sets such that $F_1 \cap F_2 = \phi$. Hence we can choose an open set O_1 such that $F_1 \subset O_1$ and $O_1 \cap F_2 = \phi$. Let $\phi_1(t)$ and $\phi_2(t)$ be continuous solutions which pass through $q_1 \in F_1$ and $q_2 \in F_2$ respectively. By Proposition 2 in [5] we see that for some $r_0>0$ and $\varepsilon_0>0$, the values of any ε-Carathèodory iterates $\phi_i(t;\xi,\varepsilon)$ at ξ of $\phi_i(t)$ belong to $V(F^*(c), r_0)$ on $[0, c]$ $(i=1,2)$ for every positive ε $(\leqq \varepsilon_0)$ and every $\xi \in [0, c]$. Then our definition implies that the relation

$$\phi_i(t;\xi,\varepsilon)=f(t)+\int_0^t g(t,s,\psi_i(s;\xi,\varepsilon))ds \tag{1}$$

holds, where the function $\psi_i(t;\xi,\varepsilon)$ associated with $\phi_i(t;\xi,\varepsilon)$ is defined by

$$\psi_i(t;\xi,\varepsilon)=\begin{cases} f(0) & \text{on } [-\varepsilon, 0] \\ \phi_i(t)=\phi_i(t;\xi,\varepsilon) & \text{on } [0,\xi] \\ \phi_i(t-\varepsilon;\xi,\varepsilon) & \text{on } (\xi, c]. \end{cases}$$

We shall show that $\phi_i(c;\xi,\varepsilon)$, $(i=1,2)$ is continuous in $\xi \in [0, c]$ for each fixed $\varepsilon \in (0, \varepsilon_0]$. The relation

$$\phi_i(c;\xi,\varepsilon)-\phi_i(c;\xi,\varepsilon)=\int_0^c \{g(c;s,\psi_i(s;\xi,\varepsilon))-g(c,s,\psi_i(s;\xi,\varepsilon))\}ds$$

is valid by (1). Moreover by the definition of ψ_i, $\psi_i(s;\xi,\varepsilon) \in V(F^*(c), r_0)$ for any s in $[0, c]$ and for any ξ in $[0, c]$. Hence, if we take $m(t,s)$ in (H3) corresponding to $l=c$ and $K=\overline{V(F^*(c), r_0)}$, $\psi_i(s;\xi,\varepsilon)$ satisfies $|g(t,s,\psi_i(s;\xi,\varepsilon))| \leqq m(t,s)$. Thus, to prove the continuity of $\phi_i(c;\xi,\varepsilon)$ in ξ, we must verify that

$$\lim_{\xi_k \to \xi} \psi_i(t;\xi_k,\varepsilon)=\psi_i(t;\xi,\varepsilon) \tag{2}$$

for almost every fixed $t \in [0, c]$. For simplicity, we put $\phi(t)=\phi_i(t)$, $\phi(t;\xi) = \phi_i(t;\xi,\varepsilon)$ and $\psi(t;\xi)=\psi_i(t;\xi,\varepsilon)$. First we shall show that (2) holds for every $t \in [0, c]$ if $\xi_k \downarrow \xi$, i.e., ξ_k tend to ξ monotonely decreasing as $k \to \infty$. Let $\xi \in [0, \varepsilon)$. For a fixed $t \in [0, \xi]$ we obtain by (1) that $\psi(t;\xi) = \psi(t;\xi_k)=\phi(t)$, so that (2) holds. We take ξ_k such that the inequality $\xi<\xi_k<\varepsilon$ holds. Then by the definition of $\psi(t;\xi_k)$, we have for $t \in [\xi, \xi+\varepsilon]$ that

$$\psi(t;\xi_k)=\begin{cases} \phi(t) & \text{when } t \in [\xi, \xi_k] \\ f(0) & \text{when } t \in (\xi_k, \varepsilon] \\ \phi(t-\varepsilon) & \text{when } t \in [\varepsilon, \xi+\varepsilon]. \end{cases}$$

Let $t \in (\xi, \varepsilon]$ be fixed. Then $\psi(t;\xi_k)=\psi(t;\xi)=f(0)$ for sufficiently large k such that $\xi<\xi_k<t$. Next let $t \in (\varepsilon, \xi+\varepsilon]$ be fixed. Then $\psi(t;\xi_k)=\psi(t;\xi) =\phi(t-\varepsilon)$ for ξ_k such that $\xi<\xi_k<\varepsilon$. Hence (2) holds for every $t \in (\xi, \xi+\varepsilon]$

and consequently for every $t \in [0, \xi+\varepsilon]$. For each fixed $t \in (\xi+\varepsilon, \xi+2\varepsilon]$ we take such ξ_k that $\xi<\xi_k<t-\varepsilon$. Since the equality

$$\begin{aligned}\psi(t;\xi_k)&=\phi(t-\varepsilon;\xi_k)\\&=f(t-\varepsilon)+\int_0^{\xi_k} g(t-\varepsilon, s, \phi(s))ds+\int_{\xi_k}^{\varepsilon} g(t-\varepsilon, s, f(0))ds\\&\quad+\int_{\varepsilon}^{t-\varepsilon} g(t-\varepsilon, s, \phi(s-\varepsilon))ds\end{aligned}$$

holds, we have

$$\begin{aligned}|\psi(t;\xi_k)-\psi(t;\xi)|&\leqq\int_{\xi}^{\xi_k}|g(t-\varepsilon, s, \phi(s))|ds+\int_{\xi}^{\xi_k}|g(t-\varepsilon, s, f(0))|ds\\&\leqq 2\int_{\xi}^{\xi_k} m(t-\varepsilon, s)ds.\end{aligned}$$

Hence we can verify (2) for each fixed $t \in (\xi+\varepsilon, \xi+2\varepsilon]$ and consequently for every $t \in [0, \xi+2\varepsilon]$. For each fixed $t \in (\xi+2\varepsilon, \xi+3\varepsilon]$ we take such ξ_k that $\xi<\xi_k<t-2\varepsilon$. Then we have

$$\begin{aligned}\psi(t;\xi_k)&=f(t-\varepsilon)+\int_0^{\xi_k} g(t-\varepsilon, s, \phi(s))ds+\int_{\xi_k}^{\varepsilon} g(t-\varepsilon, s, f(0))ds\\&\quad+\int_{\varepsilon}^{\xi_k+\varepsilon} g(t-\varepsilon, s, \phi(s-\varepsilon))ds+\int_{\xi_k+\varepsilon}^{t-\varepsilon} g(t-\varepsilon, s, \psi(s;\xi_k))ds,\end{aligned}$$

so that

$$\begin{aligned}|\psi(t;\xi_k)-\psi(t;\xi)|&\leqq\int_{\xi}^{\xi_k}|g(t-\varepsilon, s, \phi(s))|ds+\int_{\xi}^{\xi_k}|g(t-\varepsilon, s, f(0))|ds\\&\quad+2\int_{\xi+\varepsilon}^{\xi_k+\varepsilon}|g(t-\varepsilon, s, \phi(s-\varepsilon))|ds\\&\quad+\left|\int_{\xi+\varepsilon}^{t-\varepsilon}\{g(t-\varepsilon, s, \psi(s;\xi_k))-g(t-\varepsilon, s, \psi(s;\xi))\}ds\right|\\&\leqq 2\left(\int_{\xi}^{\xi_k} m(t-\varepsilon, s)ds+\int_{\xi+\varepsilon}^{\xi_k+\varepsilon} m(t-\varepsilon, s)ds\right)+\mathrm{I}\end{aligned}$$

holds. First two integrals tend to zero as $\xi_k\to\xi$. Since (2) is verified for every $t\in[\xi+\varepsilon, \xi+2\varepsilon]$, from (H2) (ii) and (H3) we see by the L.d.c.th. that $\lim_{\xi_k\to\xi}\mathrm{I}=0$. Thus we can show (2) for each fixed $t \in (\xi+2\varepsilon, \xi+3\varepsilon]$ and for every $t \in [0, \xi+3\varepsilon]$. Continuing in this fashion n times, we have for each $t \in (\xi+n\varepsilon,\ \xi+(n+1)\varepsilon]$ and ξ_k satisfying $\xi<\xi_k<t-n\varepsilon$ that

$$\begin{aligned}|\psi(t;\xi_k)-\psi(t;\xi)|&\leqq 2\int_{E_n} m(t-\varepsilon, s)\, ds\\&\quad+\left|\int_{\xi+\varepsilon}^{t-\varepsilon}\{g(t-\varepsilon, s, \psi(s;\xi_k))-g(t-\varepsilon, s, \psi(s;\xi))\}ds\right|,\end{aligned}$$

where $E_n=[\xi, \xi_k]\cup[\xi+\varepsilon, \xi_k+\varepsilon]\cup\cdots\cup[\xi+(n-1)\varepsilon, \xi_k+(n-1)\varepsilon]$. Hence by induction, we have (2) on $[0, \xi+n\varepsilon]$ for all positive integer n. If we take n so large that $\xi+n\varepsilon\geqq c$, (2) can be verified to hold for every $t \in [0, c]$.

For $\xi \in [\varepsilon, c)$, it is slightly easier to show that (2) holds at ξ for every $t \in [0, c]$ as $\xi_k \downarrow \xi$. Similarly we can show that (2) holds at $\xi \in (0, c]$ for every fixed $t \in [0, c]$ except one point ξ if $\xi_k \uparrow \xi$, i.e., ξ_k tend to ξ monotonely increasing as $k\to\infty$. Therefore $\phi_i(c;\xi,\varepsilon)$ is continuous in $\xi \in [0, c]$.

That $\phi_1(c; 0, \varepsilon)=\phi_2(c; 0, \varepsilon)$ is trivial.

We shall now define a curve

$$B_\varepsilon(\tau)=\begin{cases}\phi_1(c; c|\tau|, \varepsilon), & -1\leqq\tau\leqq 0\\ \phi_2(c; c|\tau|, \varepsilon), & 0\leqq\tau\leqq 1.\end{cases}$$

Then this curve $B_\varepsilon(\tau)$ is a continuous curve which connects two points q_1 and q_2. Hence $B_\varepsilon(\tau)$ must pass through some point d_ε in $\partial O_1\cap K$, where $d_\varepsilon=\phi_i(c; \xi, \varepsilon)$ ($i=i(\varepsilon)=1$ or 2) and $\xi=\xi(\varepsilon)$, both depending only on ε. Consequently we can find a sequence $\{\varepsilon_n\}$ such that

$$\lim_{n\to\infty}\varepsilon_n=0 \qquad \text{(monotonely decreasing)}, \ i(\varepsilon_n)=1 \text{ or } 2$$

and

$$\lim_{n\to\infty} d_n=d_0\in\partial O_1\cap K, \qquad \text{where } d_n=d_{\varepsilon_n}. \tag{3}$$

Moreover by taking a suitable subsequence of $\{\varepsilon_n\}$ if necessary, we may assume that $i(\varepsilon_n)=\text{const.}$ ($=1$ for example).

We put

$$\phi_n(t)=\phi_1(t; \xi_n, \varepsilon_n), \ \psi_n(t)=\psi_1(t; \xi_n, \varepsilon_n),$$

where $\xi_n=\xi(\varepsilon_n)$. Hence

$$\phi_n(t)=f(t)+\int_0^t g(t, s, \psi_n(s))ds \tag{4}$$

for each $t\in[0, c]$, where

$$\psi_n(t)=\begin{cases}f(0) & \text{on } [-\varepsilon_n, 0]\\ \phi_1(t)=\phi_n(t) & \text{on } [0, \xi_n]\\ \phi_n(t-\varepsilon_n) & \text{on } (\xi_n, c].\end{cases}$$

As proved in Proposition 2 in [5], we can prove that $\{\phi_n(\cdot)\}$ is relatively compact in $\mathcal{C}([0, c]; \overline{V(F^*(c), r_0)})$. Hence we can find a subsequence $\{k\}\subset\{n\}$ and $\phi_0(\cdot)\in\mathcal{C}([0, c]; \overline{V(F^*(c), r_0)})$ such that

$$\lim_{k\to\infty}\phi_k(t)=\lim_{k\to\infty}\psi_k(t)=\phi_0(t) \qquad \text{uniformly in } t\in[0, c]. \tag{5}$$

Then (4) implies that $\phi_0(t)$ is a continuous solution of (P) on $[0, c]$. By (3) and (5), we have

$$\begin{aligned}\lim_{k\to\infty} d_k&=\lim_{k\to\infty}\phi_1(c; \xi_k, \varepsilon_k)\\ &=f(c)+\lim_{k\to\infty}\int_0^c g(c, s, \psi_k(s))ds\\ &=f(c)+\int_0^c g(c, s, \phi_0(s))\,ds\\ &=\phi_0(c)=d_0\in\partial O_1\cap K,\end{aligned}$$

which contradicts the assumption that $F(c)=F_1\cup F_2$ and $F_1\cap F_2=\phi$.

Remark 1. (H5) is only needed in the proof of the latter half of the Theorem.

Remark 2. As seen in Proposition 2 in [5], if for a fixed $\varepsilon>0$, ε-Carathèodory iterates $\{\phi_i(\cdot; \xi, \varepsilon)\}_{\xi\in[0,c]}$ is relatively compact in $\mathcal{C}([0, c]; K)$, then we have $\lim_{\xi'\to\xi}\phi_i(t; \xi', \varepsilon)=\phi_i(t; \xi, \varepsilon)$ uniformly in $t\in[0, c]$.

In the Frèchet space $C[0, \alpha_M)$ with compact-open topology, we have the following result.

Corollary. *The solution family* $\mathcal{F}$ *of* (P) *is also a continuum in the Frechèt space* $C[0, \alpha_M)$.

Proof. To show this, it is sufficient to prove that $\mathcal{F}$ is a continuum in the Banach Space $C[0, c]$ for every $c \in [0, \alpha_M)$. Define a family $\mathcal{F}[\varepsilon]$ of continuous functions on $[0, c]$ by the set of ε'-Caratheòdory iterates at every $\xi \in [0, c]$ for all solution of (P) and all $\varepsilon' \in (0, \varepsilon]$, that is

$$\mathcal{F}[\varepsilon]=\{x(\cdot, \xi, \varepsilon) \in C[0, c] : x(\cdot) \in \mathcal{F}, 0 \leqq \xi \leqq c, 0<\varepsilon' \leqq \varepsilon\}.$$

Then $\mathcal{F}[\varepsilon]$ is decreasing as $\varepsilon \downarrow 0$, $\mathcal{F}[\varepsilon] \supset \mathcal{F}$ and $\mathcal{F}[\varepsilon]$ is relatively compact in $C[0, c]$ by Proposition 2 and the Remark after it in [5]. Hence if $\mathcal{F}[\varepsilon]$ is verified to be connected, the closure $\overline{\mathcal{F}[\varepsilon]}$ in $C[0, c]$ is a continuum. Now take any two functions $\phi_1(\cdot\,; \xi_1, \varepsilon_1)$ and $\phi_2(\cdot\,; \xi_2, \varepsilon_2)$ of $\mathcal{F}[\varepsilon]$, where $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are solutions of (P). Since $\phi_i(\cdot\,; \xi, \varepsilon_i)$ $(i=1, 2)$ considering ξ as parameter varies continuously in $C[0, c]$, the family $\{\phi_1(\cdot\,; \xi, \varepsilon_1)\,; \xi_1 \leqq \xi \leqq c\}$ is a continuous curve in $C[0, c]$, connecting $\phi_1(, ; \xi_1, \varepsilon_1)$ and $\phi_1(\cdot\,; c, \varepsilon_1)=\phi_1(\cdot)$. The family $\{\phi_1(\cdot\,; \xi, \varepsilon_2)\,; 0 \leqq \xi \leqq c\}$ connects $\phi_1(\cdot)$ to $\phi_1(\cdot\,; 0, \varepsilon_2)=\phi_2(\cdot\,; 0, \varepsilon_2)$ continuously. Finally, the family $\{\phi_2(\cdot\,; \xi, \varepsilon_2)\,; 0 \leqq \xi \leqq \xi_2\}$ is a continuous curve in $C[0, c]$ connecting $\phi_2(\cdot\,; 0, \varepsilon_2)$ and $\phi_2(\cdot\,; \xi_2, \varepsilon_2)$. Thus we have proved that $\mathcal{F}[\varepsilon]$ is (arcwise) connected. To complete the proof, it is sufficient to show that $\bigcap \overline{\mathcal{F}[\varepsilon]} = \mathcal{F}$. For any $\phi(\cdot) \in \bigcap \overline{\mathcal{F}[\varepsilon]}$, by definition we have a sequence $\{\phi_n(\cdot)\}$ such that $\lim_{n\to\infty} \phi_n(\cdot)=\phi(\cdot)$ in $C[0, c]$, $\phi_n(\cdot) \in \mathcal{F}[\varepsilon_n]$ $(\lim_{n\to\infty} \varepsilon_n=0)$ and

$$\phi_n(t)=f(t)+\int_0^t g(t, s, \psi_n(s))ds \qquad \text{for each } t \in [0, c]$$

where $\psi_n(\cdot)$ is the function associated with $\phi_n(\cdot)$. Since $\lim_{n\to\infty} \psi_n(t) = \phi(t)$ uniformly in $t \in [0, c]$ is verified, by letting n tend to ∞ we see that $\phi(t)$ is a solution of (P), i.e., $\phi(\cdot) \in \mathcal{F}$. Since $\bigcap \overline{\mathcal{F}[\varepsilon]} \supset \mathcal{F}$ is trivial, we see that $\bigcap \overline{\mathcal{F}[\varepsilon]} = \mathcal{F}$.

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