63. *On the Homotopy Groups of Spheres*

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The present note is concerned with the 2-component of the homotopy groups of spheres. Let π_*^n be the 2-component of the homotopy group $\pi_*(S^n)$. The groups π_{n+i}^n for $i \leq 22$ and all n have been determined in [6], [8], [9]. (If n is large, π_{n+i}^n is the 2-component of the i-th stable homotopy group of sphere spectrum and many data have been obtained by making use of the Adams spectral sequence.) In this note we are mainly concerned with the case of small n, namely unstable range.

§ 1. π_{n+i}^n for $i=23$ and 24.

The first purpose of this note is to announce the results on π_{n+i}^n for $i=23$ and 24. We completely determine the group structure of π_{n+23}^n and π_{n+24}^n for all n, by constructing the generators of π_*^n geometrically. Our method is the so-called composition method established by Toda [9]. The basic tool is the $EH\Delta$-exact sequence

$$(1.1) \qquad \cdots \longrightarrow \pi_{i+2}^{2n+1} \xrightarrow{\Delta} \pi_i^n \xrightarrow{E} \pi_{i+1}^{n+1} \xrightarrow{H} \pi_{i+1}^{2n+1} \xrightarrow{\Delta} \pi_{i-1}^n \longrightarrow \cdots$$

introduced by Whitehead and James, where E is the suspension homomorphism, H is the Hopf homomorphism and Δ is essentially the Whitehead product $[\iota_n, \ ]$. This enables us to calculate π_*^n inductively.

We now summarize the results of our calculation in the following theorem. The detailed calculations will be given in the forthcoming paper [7].

Theorem 1.2.***)

π_{n+23}^n and π_{n+24}^n are given by the table below.

(a) $\pi_{25}^2 = \{\eta_2 \circ \varepsilon_3 \circ \kappa_{11}\} \approx Z_2$

$\pi_{26}^3 = \{\bar{\alpha}\} \approx Z_4$

$\pi_{27}^4 = \{E\bar{\alpha}\} \oplus \{\nu_4 \circ \bar{\kappa}_7\} \approx Z_4 \oplus Z_8$

$\pi_{28}^5 = \{\nu_5 \circ \bar{\kappa}_8\} \oplus \{\bar{\rho}'''\} \oplus \{\phi_5\} \approx Z_8 \oplus Z_2 \oplus Z_2$

$\pi_{29}^6 = \{\nu_6 \circ \bar{\kappa}_9\} \oplus \{\bar{\rho}''\} \oplus \{\phi_6\} \oplus \{\Delta(\lambda), \Delta(\xi)\} \approx Z_8 \oplus Z_4 \oplus Z_2 \oplus (Z_8 \oplus Z_4)$

$\pi_{30}^7 = \{\nu_7 \circ \bar{\kappa}_{10}\} \oplus \{\bar{\rho}'\} \oplus \{\phi_7\} \oplus \{\bar{\kappa}_7 \circ \nu_{27} - \nu_7 \circ \bar{\kappa}_{10}\} \oplus \{\sigma' \circ \sigma_{14} \circ \mu_{21}\} \oplus \{\sigma' \circ \omega_{14}\}$
$\approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$

$\pi_{31}^8 = \{\nu_8 \circ \bar{\kappa}_{11}\} \oplus \{E\bar{\rho}'\} \oplus \{\phi_8\} \oplus \{\bar{\kappa}_8 \circ \nu_{28} - \nu_8 \circ \bar{\kappa}_{11}\} \oplus \{E\sigma' \circ \sigma_{15} \circ \mu_{22}\} \oplus \{E\sigma' \circ \omega_{15}\}$
$\oplus \{\sigma_8^2 \circ \mu_{22}\} \oplus \{\sigma_8 \circ \omega_{15}\} \oplus \{\sigma_8 \circ \eta^{*\prime}\}$
$\approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$

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***) This result was obtained independently by M. G. Barratt and M. Mahowald.

$$\pi_{32}^{9}=\{\bar{\rho}_9\}\oplus\{\nu_9\circ\bar{\kappa}_{12}\}\oplus\{\phi_9\}\oplus\{\bar{\kappa}_9\circ\nu_{29}-\nu_9\circ\bar{\kappa}_{12}\}\oplus\{\sigma_9^2\circ\mu_{23}\}\oplus\{\sigma_9\circ\omega_{16}\}$$
$$\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{33}^{10}=\{\bar{\rho}_{10}\}\oplus\{\nu_{10}\circ\bar{\kappa}_{13}\}\oplus\{\phi_{10}\}\oplus\{\bar{\kappa}_{10}\circ\nu_{30}-\nu_{10}\circ\bar{\kappa}_{13}\}\oplus\{\psi_{10}\}$$
$$\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_4$$
$$\pi_{n+23}^{n}=\{\bar{\rho}_n\}\oplus\{\nu_n\circ\bar{\kappa}_{n+3}\}\oplus\{\phi_n\}\oplus\{\psi_n\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2 \quad \textit{for } n=11,12,13,14.$$
$$\pi_{38}^{15}=\{\bar{\rho}_{15}\}\oplus\{\nu_{15}\circ\bar{\kappa}_{18}\}\oplus\{\phi_{15}\}\oplus\{\psi_{15}\}\oplus\{\bar{\varepsilon}^{*\prime}\}\oplus\{\bar{\nu}^{*\prime}\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{39}^{16}=\{\bar{\rho}_{16}\}\oplus\{\nu_{16}\circ\bar{\kappa}_{19}\}\oplus\{\phi_{16}\}\oplus\{\psi_{16}\}\oplus\{E\bar{\varepsilon}^{*\prime}\}\oplus\{E\bar{\nu}^{*\prime}\}\oplus\{\varepsilon_{16}^{*}\}\oplus\{\bar{\nu}_{16}^{*}\}$$
$$\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{40}^{17}=\{\bar{\rho}_{17}\}\oplus\{\nu_{17}\circ\bar{\kappa}_{20}\}\oplus\{\phi_{17}\}\oplus\{\psi_{17}\}\oplus\{\bar{\varepsilon}_{17}^{*}\}\oplus\{\bar{\nu}_{17}^{*}\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{41}^{18}=\{\bar{\rho}_{18}\}\oplus\{\nu_{18}\circ\bar{\kappa}_{21}\}\oplus\{\phi_{18}\}\oplus\{\psi_{18}\}\oplus\{\bar{\varepsilon}_{18}^{*}\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{n+23}^{n}=\{\bar{\rho}_n\}\oplus\{\nu_n\circ\bar{\kappa}_{n+3}\}\oplus\{\phi_n\}\oplus\{\psi_n\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2 \quad \textit{for } n=19,20,21.$$
$$\pi_{n+23}^{n}=\{\bar{\rho}_n\}\oplus\{\nu_n\circ\bar{\kappa}_{n+3}\}\oplus\{\phi_n\}\approx Z_{16}\oplus Z_8\oplus Z_2 \quad \textit{for } n=22,23.$$
$$\pi_{47}^{24}=\{\bar{\rho}_{24}\}\oplus\{\nu_{24}\circ\bar{\kappa}_{27}\}\oplus\{\phi_{24}\}\oplus\{\Delta\iota_{49}\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z$$
$$\pi_{n+23}^{n}=\{\bar{\rho}_n\}\oplus\{\nu_n\circ\bar{\kappa}_{n+3}\}\oplus\{\phi_n\}\approx Z_{16}\oplus Z_8\oplus Z_2 \quad \textit{for } n\geq 25.$$

(b) $\pi_{26}^{2}=\{\eta_2\circ\bar{\alpha}\}\approx Z_4$

$$\pi_{27}^{3}=\{\delta_3\}\oplus\{\bar{\mu}_3\circ\sigma_{20}\}\oplus\{\varepsilon'\circ\kappa_{13}\}\approx Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{28}^{4}=\{\delta_4\}\oplus\{\bar{\mu}_4\circ\sigma_{21}\}\oplus\{E\varepsilon'\circ\kappa_{14}\}\oplus\{\nu_4\circ\eta_7\circ\bar{\kappa}_8\}\oplus\{\nu_4\circ\sigma'\circ\kappa_{14}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{29}^{5}=\{\delta_5\}\oplus\{\bar{\mu}_5\circ\sigma_{22}\}\oplus\{\nu_5\circ\eta_8\circ\bar{\kappa}_9\}\oplus\{\nu_5\circ\sigma_8\circ\kappa_{15}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{30}^{6}=\{\delta_6\}\oplus\{\bar{\mu}_6\circ\sigma_{23}\}\oplus\{\nu_6\circ\sigma_9\circ\kappa_{16}\}\oplus\{\bar{\zeta}_6'\}\oplus\{\bar{\sigma}_6'\}\oplus\{\Delta(\lambda\circ\eta_{31})\}\oplus\{\Delta(\xi_{13}\circ\eta_{31})\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{31}^{7}=\{\delta_7\}\oplus\{\bar{\mu}_7\circ\sigma_{24}\}\oplus\{\nu_7\circ\sigma_{10}\circ\kappa_{17}\}\oplus\{\bar{\zeta}_7'\}\oplus\{\bar{\sigma}_7'\}\oplus\{\sigma'\circ\bar{\mu}_{14}\}\oplus\{\sigma'\circ\omega_{14}\circ\eta_{30}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{32}^{8}=\{\delta_8\}\oplus\{\bar{\mu}_8\circ\sigma_{25}\}\oplus\{\nu_8\circ\sigma_{11}\circ\kappa_{18}\}\oplus\{\bar{\zeta}_8'\}\oplus\{\bar{\sigma}_8'\}\oplus\{E\sigma'\circ\bar{\mu}_{15}\}\oplus\{E\sigma'\circ\omega_{15}\circ\eta_{31}\}$$
$$\oplus\{\sigma_8\circ\bar{\mu}_{15}\}\oplus\{\sigma_8^2\circ\eta_{22}\circ\mu_{23}\}\oplus\{\sigma_8\circ\nu_{15}\circ\kappa_{18}\}\oplus\{\sigma_8\circ\varepsilon_{15}^{*}\}\oplus\{\sigma_8\circ\eta^{*\prime}\circ\eta_{31}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{33}^{9}=\{\delta_9\}\oplus\{\bar{\mu}_9\circ\sigma_{26}\}\oplus\{\bar{\sigma}_9'\}\oplus\{\sigma_9\circ\bar{\mu}_{16}\}\oplus\{\sigma_9^2\circ\eta_{23}\circ\mu_{24}\}\oplus\{\sigma_9\circ\nu_{16}\circ\kappa_{19}\}\oplus\{\sigma_9\circ\varepsilon_{16}^{*}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{34}^{10}=\{\delta_{10}\}\oplus\{\bar{\mu}_{10}\circ\sigma_{27}\}\oplus\{\bar{\sigma}_{10}'\}\oplus\{\tilde{\varepsilon}_{10}\}\oplus\{\Delta\rho_{21}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_4\oplus Z_{32}$$
$$\pi_{35}^{11}=\{\delta_{11}\}\oplus\{\bar{\mu}_{11}\circ\sigma_{28}\}\oplus\{\bar{\sigma}_{11}'\}\oplus\{\tilde{\varepsilon}_{11}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_4$$
$$\pi_{n+24}^{n}=\{\delta_n\}\oplus\{\bar{\mu}_n\circ\sigma_{n+17}\}\oplus\{\bar{\sigma}_n'\}\oplus\{\tilde{\varepsilon}_n\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2 \quad \textit{for } n=12,13.$$
$$\pi_{38}^{14}=\{\delta_{14}\}\oplus\{\bar{\mu}_{14}\circ\sigma_{31}\}\oplus\{\bar{\sigma}_{14}'\}\oplus\{\tilde{\varepsilon}_{14}\}\oplus\{\zeta^{*}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_8$$
$$\pi_{39}^{15}=\{\delta_{15}\}\oplus\{\bar{\mu}_{15}\circ\sigma_{32}\}\oplus\{\bar{\sigma}_{15}'\}\oplus\{\tilde{\varepsilon}_{15}\}\oplus\{E\zeta^{*}\}\oplus\{\mu^{*\prime}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{40}^{16}=\{\delta_{16}\}\oplus\{\bar{\mu}_{16}\circ\sigma_{33}\}\oplus\{\bar{\sigma}_{16}'\}\oplus\{\tilde{\varepsilon}_{16}\}\oplus\{E^2\zeta^{*}\}\oplus\{E\mu^{*\prime}\}\oplus\{\mu_{16}^{*}\}\oplus\{\eta_{16}^{*}\circ\varepsilon_{32}\}\oplus\{\eta_{16}^{*}\circ\bar{\nu}_{32}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{41}^{17}=\{\delta_{17}\}\oplus\{\bar{\mu}_{17}\circ\sigma_{34}\}\oplus\{\bar{\sigma}_{17}'\}\oplus\{\mu_{17}^{*}\}\oplus\{\eta_{17}^{*}\circ\varepsilon_{33}\}\oplus\{\eta_{17}^{*}\circ\bar{\nu}_{33}\}$$
$$\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{42}^{18}=\{\delta_{18}\}\oplus\{\bar{\mu}_{18}\circ\sigma_{35}\}\oplus\{\bar{\sigma}_{18}'\}\oplus\{\Delta\sigma_{37}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_{16}$$
$$\pi_{43}^{19}=\{\delta_{19}\}\oplus\{\bar{\mu}_{19}\circ\sigma_{36}\}\oplus\{\bar{\sigma}_{19}'\}\approx Z_2\oplus Z_2\oplus Z_2$$
$$\pi_{n+24}^{n}=\{\delta_n\}\oplus\{\bar{\mu}_n\circ\sigma_{n+17}\}\approx Z_2\oplus Z_2 \quad \textit{for } n=20,21.$$

$\pi_{46}^{22}=\{\delta_{22}\}\oplus\{\bar{\mu}_{22}\circ\sigma_{39}\}\oplus\{\Delta\nu_{45}\}\approx Z_2\oplus Z_2\oplus Z_4$
$\pi_{47}^{23}=\{\delta_{23}\}\oplus\{\bar{\mu}_{23}\circ\sigma_{40}\}\oplus\{\tilde{\eta}'\}\approx Z_2\oplus Z_2\oplus Z_2$
$\pi_{48}^{24}=\{\delta_{24}\}\oplus\{\mu_{24}\circ\sigma_{41}\}\oplus\{E\tilde{\eta}'\}\oplus\{\tilde{\eta}\}\approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2$
$\pi_{49}^{25}=\{\delta_{25}\}\oplus\{\bar{\mu}_{25}\circ\sigma_{42}\}\oplus\{E\tilde{\eta}\}\approx Z_2\oplus Z_2\oplus Z_2$
$\pi_{n+24}^{n}=\{\delta_n\}\oplus\{\bar{\mu}_n\circ\sigma_{n+17}\}\approx Z_2\oplus Z_2$ *for* $n\geq 26$.

The namings of elements are given in [9], [8] and [6]. New indecomposable elements will be defined in [7] together with complete calculations.

Remark 1.3. Let ${}_2\pi_i^S$ denote the 2-component of the i-th stable homotopy group of sphere spectrum. We have shown ${}_2\pi_{23}^S=\{\bar{\rho}\}\oplus\{\nu\circ\bar{\kappa}\}\oplus\{\phi\}\approx Z_{16}\oplus Z_8\oplus Z_2$ and ${}_2\pi_{24}^S=\{\delta\}\oplus\{\bar{\mu}\circ\sigma\}$. In [7], it will be shown that ϕ and δ are decomposable although ϕ_5 and δ_3 are indecomposable. (The decomposability of ϕ was pointed out by J. Mukai.)

§ 2. Unstable periodicity. It will be useful for further calculation of π_*^n to formulate systematic phenomenon. The second purpose of this paper is to indicate the unstable version of the Adams periodicity, which was first observed by Barratt [3]. We summarize the results on the periodicity of π_*^n in the following theorem.

Theorem 2.1. (1) $\pi_{8s+4}^5\supset\{\alpha_s^{(1)}\}\approx Z_2$.

(2) π_*^n *has the following direct summands*:

(i) $(8s-1)$-*stem*; $\pi_{8s+5}^6\supset\{\alpha_s^{(2)}\}\approx Z_4$,
$\pi_{8s+6}^7\supset\{\alpha_s^{(3)}\}\approx Z_8$, $\pi_{8s+7}^8\supset\{E\alpha_s^{(3)}\}\approx Z_8$,
$\pi_{8s+8}^9\supset\{\alpha_s^{(4)}\}\approx Z_{16}$.

(ii) $8s$-*stem*;
$\pi_{8s+n}^n\supset\{\mu_{s-1,n}\circ\sigma_{8s+n-7}\}\approx Z_2$ *for* $n\geq 3$.

(iii) $(8s+1)$-*stem*;
$\pi_{8s+n+1}^n\supset\{\eta_n\circ\mu_{s-1,n+1}\circ\sigma_{8s+n-6}\}\approx Z_2$ *for* $n\geq 2$,
$\pi_{8s+n+1}^n\supset\{\mu_{s,n}\}\approx Z_2$ *for* $n\geq 3$.

(iv) $(8s+2)$-*stem*;
$\pi_{8s+n+2}^n\supset\{\eta_n\circ\mu_{s,n+1}\}\approx Z_2$ *for* $n\geq 2$.

(v) $(8s+3)$-*stem*;
$\pi_{8s+5}^2\supset\{\eta_2^2\circ\mu_{s,4}\}\approx Z_2$, $\pi_{8s+6}^3\supset\{\mu_s'\}\approx Z_4$,
$\pi_{8s+7}^4\supset\{E\mu_s'\}\approx Z_4$, $\pi_{8s+n+3}^n\supset\{\zeta_{s,n}\}\approx Z_8$ *for* $n\geq 5$.

Remark 2.2. We can not show that $\pi_{8s+4}^5\supset\{\alpha_s^{(1)}\}\approx Z_2$ is a direct summand.

The proof of Theorem 2.1 is given by making use of the d- and e-invariant of Adams [1]. The following corollaries are immediate consequences from Theorem 2.1.

Corollary 2.3.

i) $\pi_{n+2}^2\neq 0$ *if* $n\equiv 7 \mod 8$.
ii) $\pi_{n+3}^3\neq 0$ *if* $n\equiv 6, 7 \mod 8$.
iii) $\pi_{n+4}^4\neq 0$ *if* $n\equiv 7 \mod 8$.

iv) $\pi^k_{n+k} \neq 0$ *if* $k=5,6,7$ *and* $n \equiv 4,5,6 \mod 8$.

v) $\pi^8_{n+8} \neq 0$ *if* $n \equiv 4,5 \mod 8$.

Corollary 2.4.

$\pi_{n+4}(S^4) \neq 0$ *for* $n \geq 0$.

Theorem 2.1 and Corollaries are corresponding to the results of Curtis [4], which were obtained by inspection of the unstable Adams spectral sequence.

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