79. Fourier Transform of Banach Algebra Valued Functions on Group

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1. Introduction and preliminaries. Let G be a locally compact group with unit element e, and A be a complex Banach algebra with unit element 1.

Through this paper, we let Haar measure of non abelian group be left invariant, and we let $\int dx, \int dy, \cdots$, denote integration with respect to Haar measure and $m(E)$ the Haar measure of a set E.

We denote the Fourier transform $\hat{f}$ of $f \in L^1(G)$, when G is abelian, by

$$\hat{f}(\gamma)=\int_G f(x)(-x,\gamma)dx \qquad (\gamma \in \Gamma\,;\text{ the dual group of } G).$$

A well known theorem states that a functional h defined on $L^1(G)$ is a non-zero complex homomorphism if and only if

$$h(f)=\hat{f}(\gamma) \quad (f \in L^1(G)) \quad \text{for some } \gamma \in \Gamma.$$

In this paper, we give an analogue of this theorem by replacing the functions $f \in L^1(G)$ with A-valued functions on G. This is also a preliminary step to get formally a unified view about the group algebra and the representation of groups by linear transformations on a vector space, which form a Banach algebra.[1)]

Let $C_0(G\to A)$ denote the set of all A-valued continuous functions on G with compact support, and $L^1(G\to A)$ denote the completion of $C_0(G\to A)$ with respect to the norm $|||\cdot|||$, defined by

$$|||f|||=\int_G \|f(x)\|\,dx.$$

We say an A-valued function f on G is a measurable step function on G if $f(x)$ is of the form

$$f(x)=\sum_{\nu=1}^{n} a_\nu \chi_{E_\nu}(x),$$

where $a_\nu \in A$ and E_ν are measurable sets (with respect to Haar measure) with compact closure, and χ_{E_ν} are characteristic functions of E_ν.

The proofs of Proposition 1 and 2 will be given easily.

Proposition 1. *The set of all measurable step functions is dense in $L^1(G\to A)$.*

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1) L. Loomis §31 and §32.

For $f, g \in L^1(G\to A)$ we define the convolution of f and g, $f*g$, by

$$f*g(x)=\int_G f(xy)g(y^{-1})dy=\int_G f(y)g(y^{-1}x)dy.$$

And we put $f_\tau(x)=f(\tau^{-1}x)$ for $\tau, x \in G$.

Proposition 2. *$|||f_\tau|||=|||f|||$ and $(f*g)_\tau=f_\tau*g$ for each $\tau\in G$. For a fixed $f\in L^1(G\to A)$ and for any $\varepsilon>0$, there exists a neighborhood V of e in G such that $\tau\in V$ implies $|||f_\tau-f|||<\varepsilon$. In other words, the map: $\tau\to f_\tau$ is a continuous function of τ on G into $L^1(G\to A)$.*

Proposition 3. *$L^1(G\to A)$ is a Banach algebra with respect to the convolution.*

Proof. If $f, g\in C_0(G\to A)$ and if K_1 and K_2 are supports of f and g respectively, then

$$f*g(x)=\int_{K_2^{-1}} f(xy)g(y^{-1})dy=0 \qquad \text{if } x\notin K_1\cdot K_2.$$

Therefore $f*g$ has compact support. Also,

$$\|f*g(x)-f*g(x')\|\leqq\int_G \|f(xy)-f(x'y)\|\cdot\|g(y^{-1})\|\, dy$$
$$\leqq\|f_{x^{-1}}-f_{x'^{-1}}\|_\infty\cdot\int_{K_2^{-1}}\|g(y^{-1})\|\, dy.$$

Since f is uniformly continuous this implies $f*g$ is continuous. Hence if $f, g\in C_0(G\to A)$, then $f*g\in C_0(G\to A)$, and

$$|||f*g|||=\int_G\|f*g(x)\|\, dx\leqq\int_G\int_G\|f(y)\|\cdot\|g(y^{-1}x)\|\, dydx=|||f|||\cdot|||g|||.$$

If $f, g\in L^1(G\to A)$, then there exist sequences $\{f_n\}$ and $\{g_n\}$ in $C_0(G\to A)$ such that

$$\lim_{n\to\infty} f_n=f \quad \text{and} \quad \lim_{n\to\infty} g_n=g.$$

Since $f_n*g_n\in C_0(G\to A)$ for each n and since $\lim_{n\to\infty} f_n*g_n=f*g$, we have,

$$f*g\in L^1(G\to A) \quad \text{and}$$
$$|||f*g|||=\lim_{n\to\infty}|||f_n*g_n|||\leqq\lim_{n\to\infty}|||f_n|||\cdot|||g_n|||\leqq|||f|||\cdot|||g|||. \quad \text{Q.E.D.}$$

Proposition 4. *For a fixed $f\in L^1(G\to A)$ and for any $\varepsilon>0$, there exists a neighborhood V of e in G with a following property; if a measurable set $E\subset V$, then $|||m(E)^{-1}\chi_E*f-f|||<\varepsilon$.*

Remark. By Proposition 2, this inequality is equivalent to

$$|||m(E)^{-1}(\chi_E)_t*f-f_t|||<\varepsilon \qquad \text{for all } t\in G.$$

Proof. By Proposition 2, we can choose a neighborhood V of e so that $|||f_\tau-f|||<\varepsilon$ for all $\tau\in V$.

If E is a measurable set such that $E\subset V$, then we have,

$$m(E)^{-1}\chi_E*f(x)-f(x)=m(E)^{-1}\int_G\chi_E(y)f(y^{-1}x)dy-m(E)^{-1}\int_G\chi_E(y)f(x)dy$$
$$=m(E)^{-1}\int_E\{f(y^{-1}x)-f(x)\}dy,$$

and

$$\|m(E)^{-1}\chi_E*f(x)-f(x)\|\leqq m(E)^{-1}\int_E \|f_y(x)-f(x)\|\,dy,$$

so that

$$|||m(E)^{-1}\chi_E*f-f|||\leqq m(E)^{-1}\int_G\int_E \|f_y(x)-f(x)\|\,dydx$$
$$=m(E)^{-1}\int_E |||f_y-f|||\,dy<\varepsilon. \qquad \text{Q.E.D.}$$

2. Banach algebra valued homomorphism. Now we state about a Banach algebra valued homomorphism.

Let B be a Banach algebra such that $B\supset A\ni 1$, and let h be a continuous mapping of $L^1(G\to A)$ into B with following properties;

(1) $h(af+bg)=ah(f)+bh(g)$ for $a,b\in A$ and $f,g\in L^1(G\to A)$,

(2) $h(f*g)=h(f)h(g)$ for $f,g\in L^1(G\to A)$,

(3) for any $\varepsilon>0$ there exists $f_\varepsilon\in L^1(G\to A)$ such that $\|h(f_\varepsilon)-1\|_B<\varepsilon$.

Proposition 5. *For a mapping h given above, there exists a bounded continuous B-valued function φ defined on G such that*

(i) $h(f_t)=\varphi(t)h(f)$ *for $t\in G$ and $f\in L^1(G\to A)$,*

(ii) $\varphi(st)=\varphi(s)\varphi(t)$ *for $s,t\in G$,*

(iii) $a\varphi(t)=\varphi(t)a$ *for all $a\in A$ and $t\in G$.*

Proof. By the property (3), there exists $f_1\in L^1(G\to A)$ such that $h(f_1)^{-1}\in B$. Fix any $f\in L^1(G\to A)$. By Proposition 4, there exists a sequence $\{E_n\}$ of measurable sets in G such that

$$(*)\qquad \begin{aligned}&|||m(E_n)^{-1}(\chi_{E_n})_t*f_1-f_{1t}|||<1/n \quad (n=1,2,\cdots),\\ &|||m(E_n)^{-1}(\chi_{E_n})_t*f-f_t|||<1/n \quad (n=1,2,\cdots).\end{aligned}$$

Then by the continuity of h and the property (2), we have,

$$\begin{aligned}&\|m(E_n)^{-1}h(\chi_{E_{nt}})-h(f_{1t})h(f_1)^{-1}\|_B\\ &\quad\leqq\|m(E_n)^{-1}h(\chi_{E_{nt}})h(f_1)-h(f_{1t})\|_B\cdot\|h(f_1)^{-1}\|_B\\ &\quad\leqq\|h\|\cdot|||m(E_n)^{-1}\chi_{E_{nt}}*f_1-f_{1t}|||\cdot\|h(f_1)^{-1}\|_B\\ &\quad<1/n\cdot\|h\|\cdot\|h(f_1)^{-1}\|_B.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} m(E_n)^{-1}h(\chi_{E_{nt}})=h(f_{1t})h(f_1)^{-1}.$$

Putting $\varphi(t)=h(f_{1t})h(f_1)^{-1}$, we have from $(*)$,

$$\begin{aligned}\|h(f_t)-\varphi(t)h(f)\|_B&=\lim_{n\to\infty}\|h(f_t)-m(E_n)^{-1}h(\chi_{E_{nt}})h(f)\|_B\\ &\leqq\lim_{n\to\infty}\|h\|\cdot|||f_t-m(E_n)^{-1}\chi_{E_{nt}}*f|||=0.\end{aligned}$$

Hence $h(f_t)=\varphi(t)h(f)$, and since $\varphi(t)$ does not depend on the choice of sequence $\{E_n\}$ and $f\in L^1(G\to A)$, (i) is proved.

(ii) and (iii) are due to the definition of φ and (1);

$$\varphi(st)=h(f_{1st})h(f_1)^{-1}=h((f_{1t})_s)h(f_1)^{-1}=\varphi(s)\varphi(t)h(f_1)h(f_1)^{-1}=\varphi(s)\varphi(t),$$
$$\begin{aligned}a\varphi(t)&=a\varphi(t)h(f_1)h(f_1)^{-1}=ah(f_{1t})h(f_1)^{-1}=h(af_{1t})h(f_1)^{-1}\\ &=h((af_1)_t)h(f_1)^{-1}=\varphi(t)ah(f_1)h(f_1)^{-1}=\varphi(t)a, \qquad \text{if } a\in A.\end{aligned}$$

Also the boundedness and the continuity of φ are due to Proposition 2 and the continuity of h;

$$\|\varphi(t)\|_B \leqq \|h(f_{1t})\|_B \cdot \|h(f_1)^{-1}\|_B \leqq \|h\| \cdot |||f_1||| \cdot \|h(f_1)^{-1}\|_B,$$
$$\|\varphi(t)-\varphi(s)\|_B \leqq \|h(f_{1t})-h(f_{1s})\|_B \cdot \|h(f_1)^{-1}\|_B$$
$$\leqq \|h\| \cdot |||f_{1t}-f_{1s}||| \cdot \|h(f_1)^{-1}\|_B. \qquad \text{Q.E.D.}$$

Proposition 6. *For any $\varepsilon>0$, there exists a neighborhood V of e in G with the following property; if a measurable set $E\subset V$, then* $\|m(E)^{-1}h(\chi_E)-1\|_B<\varepsilon$.

Proof. Let $f_1\in L^1(G\to A)$ such that $h(f_1)^{-1}$ exists in B. In Proposition 4, if we take $\varepsilon/\|h\|\cdot\|h(f_1)^{-1}\|_B$ and f_1 in place of ε and f respectively, then we have,

$$\|m(E)^{-1}h(\chi_E)-1\|_B \leqq \|m(E)^{-1}h(\chi_E)h(f_1)-h(f_1)\|_B \cdot \|h(f_1)^{-1}\|_B$$
$$\leqq \|h\| \cdot |||m(E)^{-1}\chi_E * f_1 - f_1||| \cdot \|h(f_1)^{-1}\|_B$$
$$<\varepsilon, \qquad \text{if } E\subset V. \qquad \text{Q.E.D.}$$

Theorem. *Let h be a continuous mapping of $L^1(G\to A)$ into B with the properties* (1)~(3) *mentioned already. Then there exists a bounded continuous function φ on G into B such that*

(i) *$\varphi(x)$ commutes with all $a\in A$, for all $x\in G$,*

(ii) *$\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,y\in G$, and $\varphi(e)=1$,*

(iii) $h(f)=\int_G \varphi(x)f(x)dx.$

Conversely if a bounded continuous function φ on G into B satisfies (i) *and* (ii), *and if we define a mapping h of $L^1(G\to A)$ into B by*

$$h(f)=\int_G \varphi(x)f(x)dx \qquad (f\in L^1(G\to A)),$$

then h is continuous and satisfies (1)~(3).

Proof. Let φ be the function given in Proposition 5. Since φ satisfies the conditions (i) and (ii), we show that φ satisfies (iii).

Let E be a measurable set in G with compact closure, and for $\varepsilon>0$, let V be the neighborhood of e in G in Proposition 6.

Then there exists a decomposition of E, $E=\bigcup_{i=1}^n E_i$, such that

$$(**) \quad E_i\cap E_j=\emptyset \ (i\neq j), \qquad x_i^{-1}E_i\subset V \ (x_i\in E_i),$$

and

$$(***) \quad \left\|\sum_{i=1}^n \varphi(x_i)m(E_i)-\int_E \varphi(x)dx\right\|_B<\varepsilon.$$

Since $h(f_t)=\varphi(t)h(f)$ by Proposition 5, we have

$$h(\chi_E)=h\left(\sum_{i=1}^n \chi_{E_i}\right)=\sum_{i=1}^n h((\chi_{x_i^{-1}E_i})_{x_i})=\sum_{i=1}^n \varphi(x_i)h(\chi_{x_i^{-1}E_i}),$$

and so we have by (**), (***) and Proposition 6,

$$\left\|h(\chi_E)-\int_E \varphi(x)dx\right\|_B$$
$$\leqq \left\|h(\chi_E)-\sum_{i=1}^n \varphi(x_i)m(E_i)\right\|_B+\left\|\sum_{i=1}^n m(E_i)\varphi(x_i)-\int_E \varphi(x)dx\right\|_B$$

$$<\left\|\sum_{i=1}^{n}\varphi(x_i)m(E_i)\{m(E_i)^{-1}h(\chi_{x_i^{-1}E_i})-1\}\right\|_B+\varepsilon$$
$$\leqq\sum_{i=1}^{n}\|\varphi(x_i)\|_B\cdot m(E_i)\cdot\|m(E_i)^{-1}h(\chi_{x_i^{-1}E_i})-1\|_B+\varepsilon$$
$$<\varepsilon(\|\varphi\|_\infty\cdot m(E)+1).$$

Hence we have

$$h(\chi_E)=\int_E\varphi(x)dx=\int_G\varphi(x)\chi_E(x)dx$$

for each measurable set E in G with compact closure.

So we have, for any measurable step function $f(x)=\sum_{\nu=1}^{n}a_\nu\chi_{E_\nu}(x)$,

$$h(f)=\sum_{\nu=1}^{n}a_\nu h(\chi_{E_\nu})=\sum_{\nu=1}^{n}a_\nu\int_G\varphi(x)\chi_{E_\nu}(x)dx=\int_G\varphi(x)f(x)dx.$$

By Proposition 1, for any $f\in L^1(G\to A)$ and any $\varepsilon>0$, we can choose a measurable step function g such that $|||f-g|||<\varepsilon/2\max(\|h\|,\|\varphi\|_\infty)$. Then,

$$\left\|h(f)-\int_G\varphi(x)f(x)dx\right\|_B$$
$$\leqq\|h(f)-h(g)\|_B+\left\|\int_G\varphi(x)g(x)dx-\int_G\varphi(x)f(x)dx\right\|_B$$
$$\leqq\|h\|\cdot|||f-g|||+\|\varphi\|_\infty\cdot|||g-f|||<\varepsilon,$$

so that $h(f)=\int_G\varphi(x)f(x)dx$, for all $f\in L^1(G\to A)$.

Conversely if h is a mapping of $L^1(G\to A)$ into B defined by

$$h(f)=\int_G\varphi(x)f(x)dx\qquad(f\in L^1(G\to A)),$$

where φ is a bounded continuous function on G into B with the properties (i) and (ii), then we have,

$$\|h(f)-h(g)\|_B\leqq\|\varphi\|_\infty\cdot|||f-g|||,$$

and by (i),

$$h(af+bg)=ah(f)+bh(g),\qquad\text{if } a,b\in A.$$

Also,

$$\begin{aligned}h(f*g)&=\int_G\int_G\varphi(x)f(y)g(y^{-1}x)dydx\\&=\int_G\int_G\varphi(y)f(y)\varphi(y^{-1}x)g(y^{-1}x)dydx\\&=\int_G\varphi(y)f(y)dy\cdot\int_G\varphi(y^{-1}x)g(y^{-1}x)dx\\&=h(f)h(g)\qquad\text{for } f,g\in L^1(G\to A).\end{aligned}$$

By the continuity of φ, for any $\varepsilon>0$ we choose a neighborhood $V=V_\varepsilon$ of e in G such that $\|\varphi(x)-1\|_B<\varepsilon$ if $x\in V$.

If we put $f_\varepsilon=m(V)^{-1}\chi_V$, then we have,

$$\|h(f_\varepsilon)-1\|_B=\left\|m(V)^{-1}\int_V\{\varphi(x)-1\}dx\right\|_B\leqq m(V)^{-1}\int_V\left\|\varphi(x)-1\right\|_B dx<\varepsilon,$$

and (3) is proved. Q.E.D.

Reference

[1] L. Loomis: An Introduction to Abstract Harmonic Analysis. Van Nostrand (1953).