# 76. *On Symbols of Fundamental Solutions of Parabolic Systems*

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**Introduction.** The calculus of multiple symbols which has been developed in Kumano-go [1] enables us to construct the fundamental solution of parabolic equations only by symbol calculus (see C. Tsutsumi [4]). The purpose of the present paper is to show that a formal fundamental solution of a parabolic system has an asymptotic expansion in a class of pseudo-differential operators (§ 2) and to construct a fundamental solution with the same expansion (§ 3). The method of construction is the same as one used in C. Tsutsumi [4] for single equations.

**1. Notations and a lemma.** We shall denote by $S^m_{\rho,\delta}$ where $-\infty<m<+\infty$ and $0\leqq\delta<\rho\leqq 1$, the set of all $M\times M$ matrices $p(x,\xi)$ with components $p_{ij}(x,\xi)\in C^\infty(R^n_x\times R^n_\xi)$ which satisfy the inequality

$$|p^{(\alpha)}_{ij(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $p^{(\alpha)}_{ij(\beta)}(x,\xi)=\partial^\alpha_\xi D^\beta_x p_{ij}(x,\xi)$. We denote by $|p(x,\xi)|$ the norm of the matrix, that is,

$$|p(x,\xi)|=\sup_{0\neq y\in C^M}|p(x,\xi)y|/|y|$$

and define semi-norms $|p|_{m,k}$ by

$$|p|_{m,k}=\max_{|\alpha|+|\beta|\leqq k}\ \sup_{(x,\xi)}|p^{(\alpha)}_{(\beta)}(x,\xi)|\,\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}.$$

Then $S^m_{\rho,\delta}$ is a Fréchet space with these semi-norms. By $\mathcal{E}^0_t(S^m_{\rho,\delta})$ we denote a set of all matrices $p(t;x,\xi)\in S^m_{\rho,\delta}$ which are continuous with respect to parameter $t$ for $0\leqq t\leqq T$. By $w$-$\mathcal{E}^0_{t,s}(S^m_{\rho,\delta})$ we denote a set of all matrices $p(t,s;x,\xi)\in S^m_{\rho,\delta}$ which are continuous with respect to parameter $t$ and $s$ for $0\leqq s\leqq t\leqq T$ with weak topology of $S^m_{\rho,\delta}$ defined as follows (see H. Kumano-go and C. Tsutsumi [2]): we say $\{p_j(x,\xi)\}^\infty_{j=0}\subset S^m_{\rho,\delta}$ converges weakly to $p(x,\xi)\in S^m_{\rho,\delta}$, if $\{p_j(x,\xi)\}^\infty_{j=0}$ is a bounded set of $S^m_{\rho,\delta}$ and $p^{(\alpha)}_{j(\beta)}(x,\xi)\to p^{(\alpha)}_{(\beta)}(x,\xi)$ as $j\to\infty$ uniformly on $R^m_x\times K$ for every $\alpha,\beta$ and compact set $K\subset R^n_\xi$.

When $p_\nu(x,\xi)\in S^{m_\nu}_{\rho,\delta}$, $\nu=1,2,\cdots,j$, we denote by $p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)$ the symbol of the product $P_1P_2\cdots P_j$ of pseudo-differential operators $P_\nu=p_\nu(x,D_x)$ which has the form (see Kumano-go [1])

$$\begin{aligned}(1.1)\qquad &p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)\\ &=Os\text{-}\int\cdots\int e^{-i(y^1\eta^1+\cdots+y^{j-1}\eta^{j-1})}p_1(x,\xi+\eta^1)p_2(x+y^1,\xi+\eta^2)\cdots\\ &\qquad\cdots p_j(x+y^1+\cdots+y^{j-1},\xi)dy^1\cdots dy^{j-1}d\eta^1\cdots d\eta^{j-1}\end{aligned}$$

and we also use the following notation:

$$[p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)]_k$$

(1.2)
$$\sum_{|\alpha_2^1+\alpha_3^1+\cdots+\alpha_j^{j-1}|=k}\frac{1}{\alpha_2^1!\,\alpha_3^1!\cdots\alpha_j^{j-1}!}p_1^{(\alpha_2^1+\alpha_3^1+\cdots\alpha_j^1)}(x,\xi)p_{2,(\alpha_2^1)}^{(\alpha_3^2+\cdots+\alpha_j^2)}(x,\xi)\cdots p_{j-1,(\alpha_{j-1}^1+\cdots+\alpha_{j-1}^{j-2})}^{(\alpha_j^{j-1})}(x,\xi)p_{j,(\alpha_j^1+\cdots+\alpha_j^{j-1})}(x,\xi)$$

(cf. Nagase-Shinkai [3]). Then we have the following

**Lemma.** *When* $p_\nu(x,\xi)\in S^{m_\nu}_{\rho,\delta}$, $\nu=0,1,2,\cdots,j$, *we have*

(i) $p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)\in S^{m_1+m_2+\cdots+m_j}_{\rho,\delta}$,

(ii) *for every* $N$

(1.3)
$$p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)-\sum_{k=0}^{N-1}[p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)]_k \in S^{m_1+m_2+\cdots+m_j-(\rho-\delta)N}_{\rho,\delta}$$

*and*

(iii) (*a formula which plays a fundamental role in the proof of Theorem* 1)

(1.4)
$$[p_0(x,\xi)\circ p_1(x,\xi)\circ\cdots\circ p_j(x,\xi)]_k =\sum_{\mu=0}^{k}\sum_{|\alpha|=\mu}\frac{1}{\alpha!}p_0^{(\alpha)}(x,\xi)[p_1(x,\xi)\circ\cdots\circ p_j(x,\xi)]_{k-\mu,(\alpha)}.$$

For the proof of (i) and (ii), see Kumango-go [1]. The formula (1.4) is derived from (1.2).

**2. Asymptotic expansion of formal fundamental solutions.** We shall consider the following Cauchy problem

(2.1)
$$\begin{cases}\partial_t u+p(t;x,D_x)u=0 & \text{in } (0,T)\times R^n_x\\ u|_{t=0}=u.\end{cases}$$

under two conditions:

(i) $p(t;x,\xi)\in\mathcal{E}^0_t(S^m_{\rho,\delta})$ for $0\leqq t\leqq T$.

(ii) There exist a continuous function $\lambda(t;x,\xi)\geqslant c>0$ and positive constants $C$ and $C_{\alpha,\beta}$ which satisfy

(2.2)
$$|e_0(t,s;x,\xi)|\leqq C\exp\left[-\int_s^t\lambda(\sigma;x,\xi)d\sigma\right]\qquad\text{for } 0\leqq s\leqq t\leqq T.$$

(2.3)
$$|p^{(\alpha)}_{(\beta)}(t;x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\lambda(t;x,\xi)\qquad\text{for } 0\leqq t\leqq T.$$

Here $e_0(t,s;x,\xi)$ is the resolvent matrix of (2.1), that is,

(2.4)
$$e_0(t,s;x,\xi)=I+\sum_{j=1}^{\infty}(-1)^j\int_s^t ds_1\int_s^{s_1}ds_2\cdots\int_s^{s_{j-1}}p(s_1;x,\xi)\cdots p(s_j;x,\xi)ds_j.$$

The convergence of the right hand side of (2.4) and the estimate

$$|e_0(t,s;x,\xi)|\leqq C_1\exp[C_2(t-s)\langle\xi\rangle^m]\qquad C_1>0,\ C_2>0$$

are easily verified.

The symbol $w(t,0;x,\xi)$ of a formal fundamental solution of (2.1) is given "formally" by

(2.5)
$$w(t,s;x,\xi)=I+\sum_{j=1}^{\infty}(-1)^j\int_s^t ds_1\int_s^{s_1}ds_2\cdots\int_s^{s_{j-1}}p(s_1;x,\xi)\circ\cdots\circ p(s_j;x,\xi)ds_j,$$

and we have the following

**Theorem 1.** *There exists an asymptotic expansion*

$$w(t,s;x,\xi)\sim e_0(t,s;x,\xi)+e_1(t,s;x,\xi)+\cdots$$

*where* $e_0(t,s;x,\xi)$ *is given by* (2.4) *and for* $k\geqq 1$

$$e_k(t,s;x,\xi)$$

$$=\sum_{j=2}^{\infty}(-1)^j\int_s^t ds_1\int_s^{s_1}ds_2\cdots\int_s^{s_{j-1}}[p(s_1;x,\xi)\circ\cdots\circ p(s_j;x,\xi)]_k ds_j. \tag{2.6}$$

*For every* $k\geqq 0$ *and* $\alpha,\beta$ *we have*

$$|e_{k(\beta)}^{(\alpha)}(t,s;x,\xi)|\leqq C_{\alpha,\beta,k}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k}\omega_{k,\alpha,\beta}\exp\left[-\int_s^t\lambda(\sigma;x,\xi)d\sigma\right]. \tag{2.7}$$

*Here*

$$\omega_{k,\alpha,\beta}=\max\{\omega^2,\omega^{2k+|\alpha|+|\beta|}\}\qquad \textit{for } k\geqq 1,$$
$$\omega_{0,0,0}=1,$$
$$\omega_{0,\alpha,\beta}=\max\{\omega,\omega^{|\alpha|+|\beta|}\}\qquad \textit{for } |\alpha|+|\beta|\neq 0$$

*and*

$$\omega=\int_s^t\lambda(\sigma;x,\xi)d\sigma.$$

*Thus* $e_k(t,s;x,\xi)\in w\text{-}\mathcal{E}^0_{t,s}(S^{-k(\rho-\delta)}_{\rho,\delta})$ *for* $k=0,1,2,\cdots$.

**Proof.** Since $e_0(t,s;x,\xi)$ is the solution of the following ordinary differential equation

$$\frac{d}{dt}e_0(t,s;x,\xi)+p(t;x,\xi)e_0(t,s;x,\xi)=0 \tag{2.8}$$

with initial condition $e_0(s,s;x,\xi)=I$, differentiating (2.8) with respect to $x$ and $\xi$, we have (2.7) for the case $k=0$.

For the case $k\geqq 1$, by the formula (1.4) and the relation

$$\begin{aligned}&\frac{d}{dt}\int_s^t ds_1\int_s^{s_1}ds_2\cdots\int_s^{s_j}[p(s_1)\circ p(s_2)\circ\cdots\circ p(s_{j+1})]_k ds_{j+1}\\&=\int_s^t ds_1\int_s^{s_1}ds_2\cdots\int_s^{s_{j-1}}[p(t)\circ p(s_1)\circ\cdots\circ p(s_j)]_k ds_j\end{aligned} \tag{2.9}$$

for $j=1,2,\cdots$, we see $e_k(t,s;x,\xi)$ is the solution of the following ordinary differential equation

$$\begin{aligned}&\frac{d}{dt}e_k(t,s;x,\xi)+p(t;x,\xi)e_k(t,s;x,\xi)\\&=-\sum_{\nu=1}^{k}\sum_{|\alpha|=\nu}\frac{1}{\alpha!}p^{(\alpha)}(t;x,\xi)e_{k-\nu,(\alpha)}(t,s;x,\xi)\end{aligned} \tag{2.10}$$

with the initial condition $e_k(s,s;x,\xi)=0$. Differentiating (2.10) with respect to $x$ and $\xi$, and using (2.2) we have (2.7) for the case $k\geqq 1$.

**Remark 1.** For a scalor operator, the above conditions (2.2), (2.3) coincide with assumption (0.2), (0.3) in C. Tsutsumi [4].

**Remark 2.** When (2.1) is a Petrovskii-parabolic system, the above conditions are satisfied with

$$\lambda(t;x,\xi)=c\langle\xi\rangle^m.$$

**3. Construction of fundamental solutions.** **Theorem 2.** *Under* *the assumption* (i), (ii) *in* § 2, *we can construct a symbol* $e(t,s;x,\xi) \in w\text{-}\mathcal{E}^0_{t,s}(S^0_{\rho,\delta})$ *which satisfies the following conditions*:

(i) $e(t,0;x,D_x)$ *is the fundamental solution of* (2.1), *i.e.* $e(t,0;x,\xi)$ *satisfies the equation*

$$\begin{cases} \frac{d}{dt}e(t,0;x,\xi)+p(t;x,\xi)\circ e(t,0;x,\xi)=0 \qquad 0<t<T. \\ e(0,0;x,\xi)=I. \end{cases} \tag{3.1}$$

(ii) *For sufficiently large* $N$, *let*

$$r_N(t,s;x,\xi)=e(t,s;x,\xi)-\sum_{k=0}^{N} e_k(t,s;x,\xi). \tag{3.2}$$

*Then*

$$(t-s)^{-1}r_N(t,s;x,\xi) \in w\text{-}\mathcal{E}^0_{t,s}(S^{m-(\rho-\delta)(N+1)}_{\rho,\delta}). \tag{3.3}$$

Proof. Let

$$f_N(t,s;x,\xi)=\sum_{k=0}^{N} e_k(t,s;x,\xi)$$

and let

$$q_N(t,s;x,\xi)=-\Big(\frac{d}{dt}f_N(t,s;x,\xi)+p(t;x,\xi)\circ f_N(t,s;x,\xi)\Big).$$

Then (2.7), (2.8) and (2.10) yield the estimate

$$|q^{(\alpha)}_{N(\beta)}(t,s;x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-(\rho-\delta)(N+1)-\rho|\alpha|+\delta|\beta|}. \tag{3.4}$$

Take $N$ such that $m-(\rho-\delta)(N+1)<-n$ and let $\varphi_1(t,s;x,\xi)=q_N(t,s;x,\xi)$ and for $j=2,3,\cdots$ let

$$\begin{aligned} &\varphi_j(t,s;x,\xi) \\ &\quad=\int_s^t ds_1\int_s^{s_1} ds_2\cdots\int_s^{s_{j-2}} q_N(t,s_1;x,\xi)\circ q_N(s_1,s_2;x,\xi)\circ \\ &\qquad\qquad \cdots\circ q_N(s_{j-1},s;x,\xi)ds_{j-1}. \end{aligned} \tag{3.5}$$

Then as the proof of Proposition 3 in C. Tsutsumi [4], where the calculus of multiple symbols plays an important role, we have

$$|\varphi^{(\alpha)}_{j(\beta)}(t,s;x,\xi)| \leqq C^j_{\alpha,\beta}\frac{(t-s)^{j-1}}{(j-1)!}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(N+1)}. \tag{3.6}$$

Thus we can define $\varphi(t,s;x,\xi)$ by

$$\varphi(t,s;x,\xi)=\sum_{j=1}^{\infty}\varphi_j(t,s;x,\xi). \tag{3.7}$$

Since $\varphi(t,s;x,\xi)$ satisfies the integral equation

$$\varphi(t,s;x,\xi)=q_N(t,s;x,\xi)+\int_s^t q_N(t,\sigma;x,\xi)\circ\varphi(\sigma,s;x,\xi)d\sigma \tag{3.8}$$

and has the estimate

$$|\varphi^{(\alpha)}_{(\beta)}(t,s;x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(N+1)}, \tag{3.9}$$

we have

$$r_N(t,s;x,\xi)=\int_s^t f_N(t,\sigma;x,\xi)\circ\varphi(\sigma,s;x,\xi)d\sigma \tag{3.10}$$

and (3.3).

## References

[1] Kumano-go, H.: Pseudo-differential operators of multiple symbol and the Calderon-Vaillancourt theorem (to appear).
[2] Kumano-go, H., and Tsutsumi, C.: Complex powers of hypoelliptic pseudo-differential operators with applications. Osaka J. Math., **10**, 147–174 (1973).
[3] Nagase, M., and Shinkai, K.: Complex powers of non-elliptic operators. Proc. Japan Acad., **46**, 779–783 (1970).
[4] Tsutsumi, C.: The fundamental solution for a degenerate parabolic pseudo-differential operator. Proc. Japan Acad., **50**, 11–15 (1974).