109. *Shift Automorphism Groups of von Neumann Algebras*

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1974)

1. In the structure theory of von Neumann algebras of type III, Connes and Takesaki have treated a group G of automorphisms ($*$-preserving) of a von Neumann algebra $\mathcal{A}$ with the following property:

$$(*)\quad \begin{cases} \mathcal{A} \text{ admits a faithful semi-finite normal trace } \varphi \text{ such that} \\ \qquad\qquad \varphi\cdot g=\lambda_g\varphi \qquad\qquad (1) \\ \text{for every non trivial automorphism } g \text{ of } G \text{ and some scalar} \\ 0<\lambda_g\neq 1 \text{ depending on } g. \end{cases}$$

Especially, assume that G is a singly generated automorphism group of an abelian von Neumann algebra $\mathcal{A}$. It is proved that there exists a projection E of $\mathcal{A}$ such that

$$\{g(E)\,;\, g\in G\} \text{ is an orthogonal family} \qquad (2)$$

and

$$\sum_{g\in G} g(E)=1 \qquad (3)$$

if G satisfies the property $(*)$.

We have an interest in an automorphism group of a von Neumann algebra with such a projection.

Definition 1. Let G be an automorphism group of a von Neumann algebra $\mathcal{A}$. If there exists a projection E of $\mathcal{A}$ with (2) and (3), then G is called a *shift* and E is called a *shift projection* of G in $\mathcal{A}$. Especially, if E is a central projection, then G is called a *central shift.*

In this paper, we shall show, for a singly generated automorphism group, an elementary relation between the property $(*)$ and the notion of shift and prove the following theorem:

Theorem 2. *If G is a discrete central shift of automorphisms of a von Neumann algebra $\mathcal{A}$, then the crossed product of $\mathcal{A}$ by G is isomorphic to the tensor product $\mathcal{A}^G\otimes\mathcal{L}(L^2(G))$ of the fixed algebra $\mathcal{A}^G$ in $\mathcal{A}$ of G and the algebra $\mathcal{L}(L^2(G))$ of all bounded operators on $L^2(G)$.*

2. In order to construct the discrete crossed product of a von Neumann algebra $\mathcal{A}$ by an automorphism group G, freely acting automorphism groups play an important role.

An automorphism g of a von Neumann algebra $\mathcal{A}$ is called *freely acting* on $\mathcal{A}$

when

$$AB=g(B)A \qquad \text{for all } B \text{ in } \mathcal{A}$$

implies

$$A=0$$

([9]). An automorphism group G of $\mathcal{A}$ is called *freely acting* on $\mathcal{A}$ if $g\neq 1$ (the unit) in G is freely acting on $\mathcal{A}$.

We shall show that the property $(*)$ is stronger than the concept of free action:

Lemma 3. *Let G be an automorphism group of a von Neumann algebra $\mathcal{A}$. If G satisfies the property $(*)$, then G is freely acting on $\mathcal{A}$.*

Proof. Take $g\in G$ such that $g\neq 1$. Let F be the inner part projection of g (cf. [9]), that is, F is the maximum central projection of $\mathcal{A}$ such that $g(F)=F$ and g is an inner automorphism on $\mathcal{A}_F$. Then there exists a partial isometry V of $\mathcal{A}$ such that $V^*V=VV^*=F$ and $g(T)=V^*TV$ for each $T\in\mathcal{A}_F$. Assume that $F\neq 0$. Since φ is semifinite, it follows that there exists a nonzero projection $P\leqq F$ such as $\varphi(P)<+\infty$. By the equality (1), we have that

$$\lambda_g\varphi(P)=\varphi(g(P))=\varphi(V^*PV)=\varphi(VV^*P)=\varphi(P).$$

It implies that $\varphi(P)=0$, or $P=0$ because φ is faithful, that is a contradiction. Hence we have $F=0$, that is, g is freely acting.

Remark. Especially, if T is a fixed point of an automorphism g in G satisfying $(*)$, then $\varphi(T)=0$ or $\varphi(T)=+\infty$. Hence there is no finite trace on $\mathcal{A}$ satisfying the condition $(*)$.

Lemma 4. *Let G be a shift with a central shift projection E of a von Neumann algebra $\mathcal{A}$, then G is freely acting on the center $\mathcal{Z}$ of $\mathcal{A}$.*

Proof. Take $g\in G$ $(g\neq 1)$. Let A be an element of $\mathcal{Z}$ such as $AB=g(B)A$ for every $B\in\mathcal{Z}$. Then we have

$$Ah(E)=Agh(E) \qquad \text{for each } h\in G,$$

which implies that

$$Ah(E)=Agh(E)h(E)=0 \qquad \text{for each } h\in G.$$

Therefore $A=\sum_{h\in G}Ah(E)=0$, that is, g is freely acting on $\mathcal{Z}$. Hence G is freely acting on $\mathcal{Z}$.

As an example of a shift, there exists a finite freely acting automorphism group of an abelian von Neumann algebra (cf. [7]).

On the other hand, even if a von Neumann algebra is abelian, there exists a freely acting automorphism group which is not a shift. In fact, a countably infinite discrete group of freely acting measure preserving automorphisms of a nonatomic abelian von Neumann algebra is not a shift by Dye's result [7] and Theorem 7 in the below.

Hence, by Lemma 4, the concept of central shift is strictly stronger than free action.

For a singly generated automorphism group of an abelian von Neumann algebra, the property $(*)$ is equivalent to a trace preserving shift:

Proposition 5. *Let g be an automorphism of an abelian von Neuman algebra $\mathcal{A}$ and G the group generated by g. Then the following two statements are equivalent*:

(a) *G satisfies the property* (*).

(b) *G is a shift and $\mathcal{A}$ admits a faithful semi-finite normal trace ψ invariant under g.*

Proof. (a)$\Rightarrow$(b): It is clear by [10; Lemma 8.8] and [10; Lemma 8.9].

(b)$\Rightarrow$(a): Take $0<\lambda<1$. Define

$$\varphi(A)=\sum_{n=-\infty}^{\infty}\lambda^n\psi(Ag^n(E)) \qquad \text{for } A\in\mathcal{A},$$

where E is a shift projection of G in $\mathcal{A}$. Then we have a faithful normal trace φ on $\mathcal{A}$. Let B be a nonzero positive element in $\mathcal{A}$, then there exists an integer m such as $Bg^m(E)\neq 0$. Since ψ is a semi-finite, then we have a nonzero positive element T in $\mathcal{A}$ such as $Bg^m(E)\geqq T$ and $\psi(T)<+\infty$. We have, then,

$$\varphi(Tg^m(E))=\sum_{n=-\infty}^{\infty}\lambda^n\psi(Tg^m(E)g^n(E))=\lambda^m\psi(Tg^m(E))<+\infty,$$

so that φ is semi-finite. Finally we have

$$\begin{aligned}\varphi(g(T))&=\sum_{n=-\infty}^{\infty}\lambda^n\psi(g(T)g^n(E))\\&=\sum_{n=-\infty}^{\infty}\lambda^n\psi(Tg^{n-1}(E))\\&=\lambda\sum_{n=-\infty}^{\infty}\lambda^{n-1}\psi(Tg^{n-1}(E))=\lambda\varphi(T)\end{aligned}$$

for every $T\in\mathcal{A}$. So that we have

$$\varphi(g(T))=\lambda\varphi(T) \qquad \text{for every } T\in\mathcal{A}.$$

3. Now we shall give a brief resume of the crossed product $G\otimes\mathcal{A}$ of a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathfrak{H}$ by a discrete automorphism group G of $\mathcal{A}$ following after Connes [5] and Takesaki [10].

On the Hilbert space $L^2(G)\otimes\mathfrak{H}$, define representations I of $\mathcal{A}$ and U of G as follows,

$$(I(A)\xi)(g)=g^{-1}(A)\xi(g), \qquad g\in G,\ A\in\mathcal{A} \tag{4}$$

and

$$(U(g)\xi)(h)=\xi(g^{-1}h), \qquad g\in G,\ \xi\in L^2(G)\otimes\mathfrak{H}. \tag{5}$$

It is easily seen that I is a normal faithful representation and

$$U(g)I(A)U(g)^*=I(g(A)), \qquad A\in\mathcal{A}, g\in G. \tag{6}$$

Then the crossed product $G\otimes\mathcal{A}$ is the von Neumann algebra on $L^2(G)\otimes\mathfrak{H}$ generated by $I(\mathcal{A})$ and $U(G)$.

In [5; Proposition 1.4.6], Connes proved the following:

Theorem A. *Let $G\otimes\mathcal{A}$ be the crossed product of a von Neumann algebra $\mathcal{A}$ by a discrete automorphism group G of $\mathcal{A}$.*

(a) *The representation I is a mapping such that the matrix representation equals to $(I(A))_{g,h}=\delta_g^h g^{-1}(A)$ for $A\in\mathcal{A}$ and $g,h\in G$.*

(b) *The application e of $G\otimes\mathcal{A}$ onto $I(\mathcal{A})$ such that $e(T)=I((T)_{1,1})$ $(T\in G\otimes\mathcal{A})$ is a faithful normal expectation of $G\otimes\mathcal{A}$ onto $I(\mathcal{A})$.*

4. Now, we shall give a proof of Theorem 2. Let E be a central shift projection in $\mathcal{A}$ of G. Then, by the definition of $G\otimes\mathcal{A}$, $\{I(g(E));\ g\in G\}$ is an orthogonal family of equivalent projections in $G\otimes\mathcal{A}$ such that

$$\sum_{g\in G} I(g(E))=1.$$

This leads to that

$$G\otimes\mathcal{A}\cong(G\otimes\mathcal{A})_{I(E)}\otimes\mathcal{L}(L^2(G)).$$

Take $T\in G\otimes\mathcal{A}$. Since the shift projection is central, a direct computation implies the following equality:

$$e\{(I(E)TI(E)-I(E)e(T)I(E))^*(I(E)TI(E)-I(E)e(T)I(E))\}=0,$$

where e is the faithful expectation of $G\otimes\mathcal{A}$ onto $I(\mathcal{A})$ in Theorem A. Hence

$$I(E)TI(E)=I(E)e(T)I(E).$$

Therefore we have

$$G\otimes\mathcal{A}\cong(I(\mathcal{A}))_{I(E)}\otimes\mathcal{L}(L^2(G)).$$

We shall identify $I(\mathcal{A})$ with $\mathcal{A}$. For each $A\in\mathcal{A}$, put

$$B=\sum_{g\in G} g(A)g(E),$$

where sum exists, since E is a central shift projection of G in $\mathcal{A}$. Then $B\in\mathcal{A}^G$ and we get the following equality

$$BE=\sum_{g\in G} g(A)g(E)E=AE,$$

which implies that $\mathcal{A}_E=\mathcal{A}_E^G$.

On the other hand, the $\mathcal{A}^G$-support of E is 1. In fact, if P is a projection of $\mathcal{A}^G$ with $P\geqq E$, then

$$P=g(P)\geqq g(E)$$

and so

$$P=\sum_{g\in G} Pg(E)=\sum_{g\in G} g(E)=1.$$

Therefore A_E^G is isomorphic to $\mathcal{A}^G$. Hence we have

$$G\otimes\mathcal{A}\cong\mathcal{A}^G\otimes\mathcal{L}(L^2(G)).$$

For a finite group G of outer automorphisms of a II_1-factor $\mathcal{A}$, it holds that

$$G\otimes\mathcal{A}\cong\mathcal{A}^G\otimes\mathcal{L}(L^2(G)),$$

(cf. [1]).

5. In [2] and [3], we generalized the notions of abelian projections and of discrete von Neumann algebras. A projection $E\in\mathcal{A}$ is called *abelian over* a subalgebra $\mathcal{B}$ if $E\in\mathcal{B}^c$ and for every projection $P\in\mathcal{A}$ with $P\leqq E$, there exists a projection $Q\in\mathcal{B}$ such that $P=QE$ ([2]). A von Neumann algebra $\mathcal{A}$ is called *discrete over* $\mathcal{B}$ if there exists a projection E of $\mathcal{A}$ which is abelian over $\mathcal{B}$ and the $\mathcal{B}$-support of E is 1 ([3]).

Theorem 6. *If G is a discrete central shift automorphism group of a von Neumann algebra $\mathcal{A}$, then $\mathcal{A}$ is discrete over the fixed algebra $\mathcal{A}^G$ and furthermore $G\otimes\mathcal{A}$ is discrete over $\mathcal{A}^G$.*

Proof. In the proof of Theorem 2, we have that

$$E(G\otimes\mathcal{A})E=E\mathcal{A}E=E\mathcal{A}^GE.$$

Hence, by [4; Lemma 2], the projection E in $G\otimes\mathcal{A}$ (and in $\mathcal{A}$) is abelian over $\mathcal{A}^G$ because E belongs to $\mathcal{A}^{G'}\cap\mathcal{A}$. On the other hand, the $\mathcal{A}^G$-support of E is 1. Therefore $G\otimes\mathcal{A}$ and $\mathcal{A}$ are discrete over $\mathcal{A}^G$.

Very recently, in a mimeographed paper, Connes, Ghez, Lima, Testard and Woods defined a cohyperfinite von Neumann algebra as the following. A von Neumann algebra $\mathcal{A}$ acting on a separable Hilbert space is called *cohyperfinite* iff $\mathcal{A}\otimes I_\infty$ is hyperfinite, that is, there exists an increasing sequence $(\mathcal{N}_k)_{k=1,2,\dots}$ of type I_{2^k} subfactors of $\mathcal{A}\otimes I_\infty$ such that

$$\left(\bigcup_{k=1}^{\infty}\mathcal{N}_k\right)^{-}=\mathcal{A}\otimes I_\infty.$$

Theorem 7. *Assume that G is a discrete central shift of automorphisms of a von Neumann algebra $\mathcal{A}$. For $\mathcal{A}$ and $G\otimes\mathcal{A}$,*

(a) *If one of them is continuous, then all of them are continuous.*

(b) *If one of them is discrete, then all of them are discrete.*

(c) *If one of them is a factor, then all of them are factors.*

(d) *If one of them is cohyperfinite, then all of them are cohyperfinite.*

Proof. By Theorem 6, $\mathcal{A}$ and $G\otimes\mathcal{A}$ are discrete over $\mathcal{A}^G$. Therefore, by [6; Proposition 3] and the proof of [3; Proposition 8], we have Theorem 7.

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