105. The Hurewicz Isomorphism Theorem on Homotopy and Homology Pro-Groups

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§ 1. **Introduction.** Let (X, A, x_0) be a pair of pointed topological spaces. Let $\{\mathfrak{U}_\lambda \mid \lambda \in \Lambda\}$ be the family of all locally finite normal open covers of X such that each $\mathfrak{U}_\lambda$ has exactly one member containing x_0. Then we have an inverse system $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$ in the pro-category of the homotopy category of pairs of pointed CW complexes by taking the nerves of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\lambda \cap A$, by ordering Λ by means of refinements of covers, and by taking the homotopy classes of canonical projections. We call this inverse system the Čech system of (X, A, x_0). The Čech system of (X, A) is defined similarly by using all locally finite normal open covers of X.

We define the n-th (Čech) homotopy pro-group $\pi_n(X, A, x_0)$ to be a pro-group $\{\pi_n(X_\lambda, A_\lambda, x_{0\lambda}), \pi_n(p_{\lambda\lambda'}), \Lambda)\}$ $(n \geqq 2)$; $\pi_1(X, A, x_0) = \{\pi_1(X_\lambda, A_\lambda, x_{0\lambda}), \pi_1(p_{\lambda\lambda'}), \Lambda\}$ is considered as a pro-object in the category of pointed sets and base-point preserving maps.

The n-th (Čech) homology pro-group $H_n(X, A)$ with coefficients in the additive group of integers is defined similarly by using the Čech system of (X, A). Since $\{\mathfrak{U}_\lambda \mid \lambda \in \Lambda\}$ described above is cofinal in the family of all locally finite normal open covers of X, the inverse system $\{H_n(X_\lambda, A_\lambda), H_n(p_{\lambda\lambda'}), \Lambda\}$ is isomorphic to $H_n(X, A)$ in the category of pro-groups. Hence, the set of the Hurewicz homomorphisms $\Phi_n(X_\lambda, A_\lambda, x_{0\lambda}) : \pi_n(X_\lambda, A_\lambda, x_{0\lambda}) \to H_n(X_\lambda, A_\lambda)$ for $\lambda \in \Lambda$ determines a morphism $\Phi_n(X, A, x_0) : \pi_n(X, A, x_0) \to H_n(X, A)$ in the category of pro-groups, which we shall call the Hurewicz morphism.

A subspace A of a space X is said to be P-embedded in X if every locally finite normal open cover of A has a refinement which can be extended to a locally finite normal open cover of X. If A is P-embedded in X, $\{(A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'} \mid (A_{\lambda'}, x_{0\lambda'})], \Lambda\}$, which is obtained from the Čech system of (X, A, x_0), is isomorphic to the Čech system of (A, x_0). A pro-group $G = \{G_\lambda, \phi_{\lambda\lambda'}, \Lambda\}$ is a zero-object, $G = 0$ in notation, if G is isomorphic to a pro-group consisting of a single trivial group, or equivalently, if for each $\lambda \in \Lambda$ there is $\lambda' \in \Lambda$ with $\lambda < \lambda'$ such that $\phi_{\lambda\lambda'} = 0$.

In this paper we shall establish the following analogue of the Hurewicz isomorphism theorem.

Theorem 1. *Let (X, A, x_0) be a pair of pointed, connected, topological spaces such that $\pi_k(X, A, x_0) = 0$ for k with $1 \leqq k \leqq n$ $(n \geqq 1)$. Then*

$H_k(X,A)=0$ for $1\leqq k\leqq n$. If A is P-embedded in X and $\pi_1(A,x_0)=0$ then the Hurewicz morphism $\Phi_{n+1}(X,A,x_0):\pi_{n+1}(X,A,x_0)\to H_{n+1}(X,A)$ is an isomorphism.

For the absolute Herewicz isomorphism theorem, its analogue was proved by K. Kuperberg [2] for compact metric spaces,[1] and for the relative Hurewicz isomorphism theorem its analogue was proved by T. Porter [6] for movable pairs of metric compacta with a certain condition. These results, however, are concerned with the limit groups of homotopy and homology pro-groups, but they are direct consequences of our Theorem 1.

§ 2. **Some lemmas.** Let $\mathfrak{K}$ be a category. Let $X=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $Y=\{Y_\mu, q_{\mu\mu'}, M\}$ be inverse systems (over directed sets) in $\mathfrak{K}$. A map of inverse systems, or simply a system map, from X to Y consists of a map $\phi: M\to\Lambda$ and a collection $\{f_\mu \mid \mu\in M\}$ of morphisms $f_\mu: X_{\phi(\mu)}\to Y_\mu$ such that for every $\mu, \mu'\in M$ with $\mu<\mu'$ there is $\lambda\in\Lambda$ such that $\phi(\mu)$, $\phi(\mu')<\lambda$ and $f_\mu p_{\phi(\mu)\lambda}=q_{\mu\mu'}f_{\mu'}p_{\phi(\mu')\lambda}$. Two system maps $f=\{\phi, f_\mu, M\}$ and $g=\{\psi, g_\mu, M\}$ from X to Y is called equivalent if for each $\mu\in M$ there is $\lambda\in\Lambda$ such that $\phi(\mu)<\lambda, \psi(\mu)<\lambda$ and $f_\mu p_{\phi(\mu)\lambda}=g_\mu p_{\psi(\mu)\lambda}$. The equivalence class containing f is denoted by $[f]$. There is a category whose objects are inverse systems in $\mathfrak{K}$ and whose morphisms are equivalence classes of system maps. It is called the pro-category of $\mathfrak{K}$ and is denoted by pro $(\mathfrak{K})$.

If Λ' is a cofinal subset of Λ, then $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is isomorphic to $\{X_\lambda, p_{\lambda\lambda'}, \Lambda'\}$ in pro $(\mathfrak{K})$.

Lemma 1. *Let $(\Lambda, <)$ be a directed set with order $<$. Let $\prec$ be another order in Λ such that* (i) $\lambda\prec\lambda'\Rightarrow\lambda<\lambda'$, (ii) $\forall\lambda\in\Lambda, \exists\mu\in\Lambda: \lambda\prec\mu$, *and* (iii), $\lambda<\lambda'\prec\mu'<\mu\Rightarrow\lambda\prec\mu$. *Then $(\Lambda, \prec)$ is also a directed set and any inverse system $\{X_\lambda, p_{\lambda\lambda'}, (\Lambda, \prec)\}$ in $\mathfrak{K}$ is isomorphic to $\{X_\lambda, p_{\lambda\lambda'}, (\Lambda, \prec)\}$ in* pro $(\mathfrak{K})$.

Proof. The first part is obvious. For any $\lambda\in\Lambda$, we choose an element $\phi(\lambda)$ of Λ so that $\lambda\prec\phi(\lambda)$, and let us define $f_\lambda: X_{\phi(\lambda)}\to X_\lambda$ by $f_\lambda=p_{\lambda\phi(\lambda)}$. On the other hand, let us put $g_\lambda=1: X_\lambda\to X_\lambda$. Then $f=\{\phi, f_\lambda, (\Lambda, \prec)\}: \{X_\lambda, p_{\lambda\lambda'}, (\Lambda, <)\}\to\{X_\lambda, p_{\lambda\lambda'}, (\Lambda, \prec)\}$ and $g=\{1, g_\lambda, (\Lambda, <)\}: \{X_\lambda, p_{\lambda\lambda'}, (\Lambda, \prec)\}\to\{X_\lambda, p_{\lambda\lambda'}, (\Lambda, <)\}$ are system maps, and $[f][g]=1$, $[g][f]=1$.

Lemma 2. *Let $X=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $Y=\{Y_\lambda, q_{\lambda\lambda'}, \Lambda\}$ be inverse systems in $\mathfrak{K}$ over the same directed set Λ. Suppose that for each $\lambda\in\Lambda$ there exists a morphism $f_\lambda: X_\lambda\to Y_\lambda$ and for any $\lambda, \mu\in\Lambda$ with $\lambda<\mu$ there exists $\psi_{\lambda\mu}: Y_\mu\to X_\lambda$ such that*

$$(1) \qquad p_{\lambda\mu}=\psi_{\lambda\mu}f_\mu, \qquad f_\lambda\psi_{\lambda\mu}=q_{\lambda\mu}.$$

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1) The proof in [4, §6] for topological spaces is incorrect. Our proof of Theorem 1 is its rectification.

Then $f=\{1, f_\lambda, \Lambda\}$ *defines a system map from* X *to* Y *which induces an isomorphism in pro* $(\mathfrak{K})$.

Proof. For $\kappa, \lambda, \mu, \nu \in \Lambda$ such that $\kappa<\lambda<\mu<\nu$, we have

(2) $$\psi_{\kappa\lambda}q_{\lambda\nu}=\psi_{\kappa\mu}q_{\mu\nu}=p_{\kappa\lambda}\psi_{\lambda\mu}q_{\mu\nu},$$

since $\psi_{\kappa\mu}q_{\mu\nu}=p_{\kappa\lambda}p_{\lambda\mu}\psi_{\mu\nu}=p_{\kappa\lambda}\psi_{\lambda\mu}q_{\mu\nu}=\psi_{\kappa\lambda}q_{\lambda\mu}q_{\mu\nu}$. For each $\lambda\in\Lambda$ let us choose an element $\alpha(\lambda)\in\Lambda$ so that $\lambda<\alpha(\lambda)$, and define $g_\lambda: Y_{\alpha(\lambda)}\to X_\lambda$ by $g_\lambda=\psi_{\lambda,\alpha(\lambda)}$. If $\kappa<\kappa'<\mu<\nu$ and $\kappa<\lambda<\lambda'<\mu<\nu$, then by (1) and (2) we have $\psi_{\kappa\kappa'}q_{\kappa'\nu}=\psi_{\kappa\mu}q_{\mu\nu}=\psi_{\kappa\lambda'}q_{\lambda'\nu}=p_{\kappa\lambda}\psi_{\lambda\lambda'}q_{\lambda'\nu}$. Hence $g=\{\alpha, g_\lambda, \Lambda\}$ defines a system map from Y to X. Since $[f][g]=1$ and $[g][f]=1$, this completes the proof of Lemma 2.

Lemma 3. *Let* $p_{i+1,i}: (X_i, A_i, x_i)\to(X_{i+1}, A_{i+1}, x_{i+1})$, $0\leqq i<n$, *be continuous maps of pairs of pointed connected simplicial complexes such that*

$$\pi_{k+1}(p_{k+1,k})=0: \pi_{k+1}(X_k, A_k, x_k)\to\pi_{k+1}(X_{k+1}, A_{k+1}, x_{k+1})$$

for $0\leqq k<n$. *Then there is a continuous map* $\psi: (X_0, X_0^n\cup A_0, x_0)\to(X_n, A_n, x_n)$ *such that*

$$\psi j\simeq p_{n,n-1}\cdots p_{10}: (X_0, A_0, x_0)\to(X_n, A_n, x_n),$$

where X_0^k *is the* k*-skeleton of* X_0 *and* $j: (X_0, A_0, x)\to(X_0, X_0^n\cup A_0, x_0)$ *is the inclusion map. Moreover, if* $\pi_1(p_{10}|(A_0, x_0))=0$ *and* $\pi_1(p_{10}|(X_0, x_0))=0$, *then* ψ *can be chosen so that* $\psi(X_0^1)=x_n$.

Proof. In what follows, maps are continuous. Assume that $\pi_1(p_{10}|(A_0, x_0))=0$ and $\pi_1(p_{10}|(X_0, x_0))=0$. Putting $L_0=X_0^0\times I\cup X_0\times 0$ and $L_k=(X_0^k\cup A_0)\times I\cup X_0\times 0$ for $1\leqq k\leqq n$, where $I=[0,1]$, let us construct maps $\chi_k: L_k\to X_k, k=0, 1, \cdots, n$ with the following properties.

(3) $\chi_0(x, 0)=x$ for $x\in X_0$, $\chi_0(A_0^0\times I)\subset A_0$;

(4) $\chi_1(x, 1)=x_1$ for $x\in X_0^1$, $\chi_1(A_0\times I)\subset A_1$;

(5) $\chi_k|L_{k-1}=p_{k,k-1}\chi_{k-1}$ for $k\geqq 1$;

(6) $\chi_k(x, 1)\in A_k$ for $x\in X_0^k$, $k\geqq 0$.

First, let χ_0 be defined over $X_0\times 0$ by (3). For $x\in X_0^0$ let $\chi_0(x, t)$ be a path from x to x_0 so that it lies in A_0 if $x\in A_0^0$. Next, let E^1 be a 1-simplex in X_0 (resp. A_0). Then χ_0 defines a map α from $(E^1\times 0\cup \dot{E}^1\times I, \dot{E}^1\times 1)$ to (X_0, x_0) (resp. (A_0, x_0)). Since $\pi_1(p_{10}|(X_0, x_0))=0$ and $\pi_1(p_{10}|(A_0, x_0))=0$, $p_{10}\alpha$ is homotopic in X_1 (resp. A_1) relative to $\dot{E}^1\times 1$ to a map from $E^1\times 0\cup\dot{E}^1\times I$ to x_1. This homotopy yields an extension β of $p_{10}\alpha$ over $E^1\times I$ such that $\beta(E^1\times 1)=x_1$ and $\beta(E^1\times I)\subset A_1$ if $E^1\subset A_0$. Hence $p_{10}\chi_0$ is extended to a map $\chi_1: L_0\cup X_0^1\times I\to X_1$ such that $\chi_1(X_0^1\times 1)=x_1$ and $\chi_1(A_0^1\times I)\subset A_1$. Then by the homotopy extension theorem χ_1 is extended over $L_0\cup(X_0^1\times I)\cup(A_0\times I)$ such that $\chi_1(A_0\times I)\subset A_1$. Since $L_1=L_0\cup(X_0^1\times I)\cup(A_0\times I)$, the extended map χ_1 satisfies (4), (5) and (6) with $k=1$.

For $k\geqq 2$, suppose that χ_{k-1} has been constructed. Let E^k be a k-simplex in X_0-A_0. Then χ_{k-1} induces a map α from $(E^k\times 0\cup\dot{E}^k\times I,$

$\dot{E}^k \times 1, x\times 1)$ to $(X_{k-1}, A_{k-1}, x_{k-1})$ where $x \in \dot{E}^k \cap X_0^0$. Since $\pi_k(p_{k,k-1}) = 0$, $p_{k,k-1}\alpha$ is homotopic relative to $\dot{E}^k \times 1$ to a map from $E^k \times 0 \cup \dot{E}^k \times I$ to A_k. This homotopy yields an extension β of $p_{k,k-1}\alpha$ over $E^k \times I$ such that $\beta(E^k \times 1) \subset A_k$. Hence we can find χ_k satisfying (5) and (6). Therefore by induction on k we can find χ_k satisfying (5) and (6) for all k with $2 \leqq k \leqq n$. Here we note that $\chi_n(x, 1) = x_n$ for $x \in X_0^1$.

Finally, by the homotopy extension theorem there is a map $\theta : X_0 \times I \to X_n$ such that $\theta | L_n = \chi_n$. Let us put $\psi(x) = \theta(x, 1)$ for $x \in X_0$. Then ψ has the desired properties. This proves the second part of Lemma 3.

The first part is proved similarly; it is essentially due to Mardešić [3].

§ 3. Proof of Theorem 1. Assume $\pi_1(A, x_0) = 0$ and $\pi_1(X, A, x_0) = 0$. Then by the exactness of the sequence of homotopy pro-groups (cf. [3], [5]) we have $\pi_1(X, x_0) = 0$. Hence for each $\lambda \in \Lambda$ there is $\mu \in \Lambda$ which admits a sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ in Λ such that $\lambda < \lambda_n < \cdots < \lambda_0 < \mu$ and $p_{\lambda_{i+1}\lambda_i} : (X_{\lambda_i}, A_{\lambda_i}, x_{0\lambda_i}) \to (X_{\lambda_{i+1}}, A_{\lambda_{i+1}}, x_{0\lambda_{i+1}})$, $i = 0, 1, \cdots, n-1$ satisfy the conditions in Lemma 3 (with the subscripts i there replaced by λ_i). In such a case we write $\lambda \prec \mu$. Then by Lemmas 1 and 3 the inverse system $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], (\Lambda, <)\}$ is isomorphic to the Čech system of (X, A, x_0) and for $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ there exists a map $\phi_{\lambda\mu} : (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu}) \to (X_\lambda, A_\lambda, x_{0\lambda})$ such that $p_{\lambda\mu} \simeq \phi_{\lambda\mu} j_\mu : (X_\mu, A_\mu, x_{0\mu}) \to (X_\lambda, A_\lambda, x_{0\lambda})$ and $\phi_{\lambda\mu}(X_\mu^n \cup A_\mu) \subset A_\lambda$, $\phi_{\lambda\mu}(X_\mu^1) = x_{0\lambda}$, where $j_\mu : (X_\mu, A_\mu, x_{0\mu}) \to (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu})$ is the inclusion map. Let us now construct the quotient space $Y_\mu = X_\mu / X_\mu^1$ and put $B_\mu = (X_\mu^n \cup A_\mu)/X_\mu^1$; let $g_\mu : (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu}) \to (Y_\mu, B_\mu, y_{0\mu})$ be the quotient map. Then there is a map $\psi_{\lambda\mu} : (Y_\mu, B_\mu, y_{0\mu}) \to (X_\lambda, A_\lambda, x_{0\lambda})$ such that $\phi_{\lambda\mu} = \psi_{\lambda\mu} g_\mu$. It is to be noted that $\pi_k(Y_\mu, B_\mu, y_{0\mu}) = 0$ for $1 \leqq k \leqq n$, $\pi_1(B_\mu, y_{0\mu}) = 0$, and (Y_μ, B_μ) is a pair of connected CW complexes. Thus, the usual Hurewicz homomorphism

$$\Phi_{n+1}(Y_\mu, B_\mu, y_{0\mu}) : \pi_{n+1}(Y_\mu, B_\mu, y_{0\mu}) \to H_{n+1}(Y_\mu, B_\mu)$$

is an isomorphism. If we put

$$\theta_{\lambda\mu} = \pi_{n+1}(\psi_{\lambda\mu}) \circ \Phi_{n+1}(Y_\mu, B_\mu, y_{0\mu})^{-1} \circ H_{n+1}(g_\mu j_\mu) : H_{n+1}(X_\mu, A_\mu) \to \pi_{n+1}(X_\lambda, A_\lambda, x_{0\lambda}),$$

then we have

$$\theta_{\lambda\mu} \circ \Phi_{n+1}(X_\mu, A_\mu, x_{0\mu}) = \pi_{n+1}(p_{\lambda\mu}),$$
$$\Phi_{n+1}(X_\lambda, A_\lambda, x_{0\lambda}) \circ \theta_{\lambda\mu} = H_{n+1}(p_{\lambda\mu}).$$

Therefore, by Lemma 2, $\{1, \Phi_{n+1}(X_\lambda, A_\lambda, x_{0\lambda}), (\Lambda, \prec)\}$ defines an isomorphism from $\{\pi_{n+1}(X_\lambda, A_\lambda, x_{0\lambda}), \pi_{n+1}(p_{\lambda\lambda'}), (\Lambda, \prec)\}$ to $\{H_{n+1}(X_\lambda, A_\lambda), H_{n+1}(p_{\lambda\lambda'}), (\Lambda, \prec)\}$. Thus the second part of Theorem 1 is proved.

The first part is proved similarly (but more easily since $H_k(X_\mu, X_\mu^n \cup A_\mu) = 0$ for $1 \leqq k \leqq n$).

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