# 137. On Isolated Components of Elements in a Compactly Generated l-Semigroup

By Derbiau F. HSU*)

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

Recently, Murata and Hsu [2], [3] have presented analogous results of [4] for elements of an $l$-semigroup with a compact generator system. In [1] by defining an isolated component, authors have done a continued work of [4] to investigate the ideals which can be represented as the intersection of a finite number of $f$-primary ideals. The purpose of this note is to generalize results in [1] to elements in a compactly generated $l$-semigroup with a compact generator system.

Let $L$ be a $cl$-semigroup with the following conditions as same as in [2], [3]:

($\alpha$) If $M$ is a $\varphi$-system with kernel $M^*$, and if for any element $a$ of $L, M$ meets $\Sigma(a)$, then $M^*$ meets $\Sigma(a)$.

($\beta$) For any $\varphi$-primary element $q$ of $L$, $q:q=e$. Moreover, if for any $\varphi$-system $M$, $\Sigma(r(q))$ meets $M$, then $\Sigma(q)$ meets $M$.

Throughout this note, we shall denote $r(a)$ as the $\varphi$-radical of an element $a$ of $L$. Other terms are as same as in [2], [3].

**1. Isolated components. Definition 1.1.** Let $a$ be an element of $L$ and $M$ be a $\varphi$-system. The isolated component $a(M)$ of $a$ determined by $M$ will be defined as the supremum of all $\{a:m\}$, $m$ runs over $M$, when $M$ is not empty. $a(M)$ is defined to be $a$, when $M$ is empty.

As in [3], we have assumed that there is such element $x$ for any $a \in L$ and any $u \in \Sigma$ with $\varphi(u)\varphi(x) \leqq a, x \in \Sigma$. Then there exists such element $a:m$ in $L$ and it can be seen from (3.2) in [3] that $a \leqq a(M)$.

**Lemma 1.2.** *Let $M^*$ be any kernel of a $\varphi$-system $M$. If $x \in \Sigma(a(M))$, there exists an element $m^*$ of $M^*$ such that $\varphi(m^*)\varphi(x)$ is less than $a$.*

**Proof.** Since $x \in \Sigma(a(M))$, we have $x \leqq a(M) = \sup\left\{\bigvee_{m \in M} N_m\right\}$, when $M$ is not empty (if $M$ is empty, it is trivial), where $N_m$ is the set of the compact elements $u$'s such that $\varphi(m)\varphi(u) \leqq a$, and $\bigvee$ denotes the set-theoretic union. Then we can find a finite number of elements $x_i$ of $\bigvee_{m \in M} N_m$ such that $x \leqq \bigcup_{i=1}^{n} x_i$. Suppose that $x_i \in N_{m_i}$, then $\varphi(m_i)\varphi(x_i) \leqq a$, $x \leqq \bigcup_{i=1}^{n} x_i \leqq \bigcup_{i=1}^{n} \varphi(x_i)$, $\varphi(x) \leqq \bigcup_{i=1}^{n} \varphi(x_i)$. Moreover, we can find $m_i^*$ of $M^*$

*) Department of Mathematics, National Central University, Chung-Li, Taiwan.

such that $m_i^* \leqq \varphi(m_i)$ for $i=1,2,\cdots,n$. Take an element $m^*$ of $M^*$ such that $m^* \leqq \prod_{i=1}^{n} m_i^*$. Then we have $\varphi(m^*) \leqq \varphi(m_i)$ and $\varphi(m^*)\varphi(x) \leqq \bigcup_{i=1}^{n} \varphi(m^*)\varphi(x_i) \leqq \bigcup_{i=1}^{n} \varphi(m_i)\varphi(x_i) \leqq a$ as desired.

**Proposition 1.3.** *An element $q$ of $L$ is $\varphi$-primary if and only if, for any $\varphi$-system $M$, either $q(M)=q$ or $q(M)=e$ holds.*

**Proof.** Suppose that $q$ is $\varphi$-primary and there exists a $\varphi$-system $M$ such that $q(M) \neq q$. Then we have $q \lneqq q(M)$. This implies that there exists $x$ in $\Sigma(q(M))$ but not in $\Sigma(q)$. By (1.2), there exists an element $m^*$ of $M^*$ such that $\varphi(m^*)\varphi(x)$ is less than $q$. Because $q$ is $\varphi$-primary and $x \not\leqq q$, we have $m^* \leqq r(q)$. This means $M$ meets $\Sigma(r(q))$. By condition ($\beta$), $M$ meets $\Sigma(q)$. Then there exists $m$ in $M$ and in $\Sigma(q)$. It follows that $\bigcap_{q' \in \Sigma(q)} (q : q') = e$ (since $q : q = e$), then $q : q' = e$ for all $q'$ in $\Sigma(q)$. We have then $q : m = e$ and then $q(M) = e$.

Conversely, suppose for any $\varphi$-system $M$, either $q(M)=q$ or $q(M)=e$ and $q$ is not $\varphi$-primary. Then there exists $a \not\leqq q, b \not\leqq r(q)$ such that $\varphi(a)\varphi(b) \leqq q, a, b \in \Sigma$. Since $b \not\leqq r(q) = \inf_i p_i$, where $p_i$ is $\varphi$-prime element and greater than $q$, there exists a $\varphi$-prime element $p$ such that $p \geq q$ and $b \not\leqq p$. Then $M = \Sigma'(p)$ is a $\varphi$-system. By the fact $q : b = \sup\{x \mid \varphi(x)\varphi(b) \leqq q\}$ and $\varphi(a)\varphi(b) \leqq q$, we have $a \leqq q : b$. Moreover, we know that $b$ is in $M$. Then $a \leqq q : b \leqq \bigcup_{m \in M} \{q : m\} = q(M)$. Therefore $q \lneqq q(M)$. Then we have $q(M)=e$. Since $b$ is in $\Sigma(q(M))$, by (1.2), there exists an element $m^*$ in $M^*$ such that $\varphi(m^*)\varphi(b)$ is less than $q$. Now $\varphi(m^*)\varphi(b) \leqq q \leqq p$. We have $m^* \leqq p$ or $b \leqq p$. But both are impossible. Then $q$ is $\varphi$-primary as desired.

If an element $a$ has an $\varphi$-primary decomposition, then the isolated component of $a$ can be expressed in terms of its $\varphi$-primary components.

**Theorem 1.4.** *Let $a$ be an element of $L$ and $M$ be an $\varphi$-system. Suppose that $a = q_1 \cap q_2 \cap \cdots \cap q_n$, where $q_i$ is $\varphi$-primary. If $\Sigma(r(q_i))$ meets $M$ for $n^*+1 \leqq i \leqq n$ but not for $1 \leqq i \leqq n^*$, then we have $a(M) = q_1 \cap q_2 \cap \cdots \cap q_{n^*}$. If $\Sigma(r(q_i))$ meets $M$ for $1 \leqq i \leqq n$,* then $a(M)=e$.

**Proof.** If $M$ is empty, the theorem is trivial. So, we assume that $M$ is not empty. Let $x \leqq a(M) = \sup_{m \in M} \{a : m\}$. As the proof of (1.3), we get $m^* \in M^* \subseteq M$ such that $\varphi(m^*)\varphi(x) \leqq a = q_1 \cap q_2 \cap \cdots \cap q_n$, where $M^*$ is the kernel of $M$. Then we have $\varphi(m^*)\varphi(x) \leqq q_i, i=1,2,\cdots,n$. For $1 \leqq i \leqq n^*, M$ does not meet $\Sigma(r(q_i))$. It means that $m^* \not\leqq r(q_i)$ for all $m^*$ in $M^*$. Since $q_i$ is $\varphi$-primary, we have $x \leqq q_i$, for $i=1,2,\cdots,n^*$ and hence $a(M) \leqq q_1 \cap q_2 \cap \cdots \cap q_{n^*}$.

For $n^*+1 \leqq j \leqq n$, $\Sigma(r(q_j))$ meets $M$ and hence by ($\alpha$) and ($\beta$), $\Sigma(r(q_j))$ meets $M^*$. Since $M^*$ is a $\mu$-system, for $m_{n^*+1} \in \Sigma(q_{n^*+1}) \cap M^*$ and $m_{n^*+2}$

$\in \Sigma(q_{n^*+2})\cap M^*$, there exists $m'_{n^*+2}\in M^*$ such that $m'_{n^*+2}\leqq m_{n^*+1}m_{n^*+2}$ $\in \Sigma(q_{n^*+1})\cap\Sigma(q_{n^*+2})\cap M^*$. Similarly, there exists $m'_{n^*+3}\in M^*$ and $m'_{n^*+3}$ $\leqq m'_{n^*+2}m_{n^*+3}\in\Sigma(q_{n^*+1})\cap\Sigma(q_{n^*+2})\cap\Sigma(q_{n^*+3})\cap M^*$ for $m_{n^*+3}\in\Sigma(q_{n^*+3})\cap M^*$. Continuing in this way, we obtain after a finite number of steps an element $m'_n$ such that $m'_n\in\Sigma(q_{n^*+1})\cap\cdots\cap\Sigma(q_n)\cap M^*$. Hence by condition ($\beta$), we have $q_j:m'_n=e$. On the other hand, we have $q_1\cap\cdots\cap q_{n^*}$ $\leqq(q_1\cap\cdots\cap q_{n^*}):m'_n=(q_1\cap q_2\cap\cdots\cap q_n):m'_n=a:m'_n\leqq a(M)$. Then we can conclude that $a(M)=q_1\cap q_2\cap\cdots\cap q_{n^*}$.

If for $1\leqq i\leqq n$, $\Sigma(r(q_i))$ meets $M$, then the above proof shows that there exists $m'_n$ such that $m'_n\in\Sigma(q_1)\cap\cdots\cap\Sigma(q_n)\cap M^*$ and then $(q_1\cap\cdots\cap q_n):m'_n=e$, hence we have $a(M)=e$. This completes the proof.

Combining this theorem with (1.3), we can conclude that if $q$ is $\varphi$-primary element in $L$, then $q(M)$ is $e$ or $q$ according as $\Sigma(r(q))$ meets or does not meet $M$, where $M$ is a $\varphi$-system.

From (1.4), we see immediately the following corollary:

**Corollary 1.5.** *A decomposable element of L has at most a finite number of isolated components.*

**2. Isolated set. Lemma 2.1.** *Suppose that an element a of L has an $\varphi$-primary decomposition: $a=q_1\cap q_2\cap\cdots\cap q_n$. Then any $\varphi$-prime element p which is greater than a must be greater than at least one of the $q_i$.*

**Proof.** If $p=e$, the lemma is trivial. Therefore we may suppose there exists a $\varphi$-prime element $p\neq e$ such that $a\leqq p$ and $q_i\not\leqq p$ for $1\leqq i$ $\leqq n$. We can see that $\Sigma'(p)$ is a $\varphi$-system. Then we have $\Sigma'(p)$ meets $\Sigma(q_i)$ for all $i=1,2,\cdots,n$. It follows that $\Sigma'(p)$ meets $\Sigma(r(q_i))$ for all $i=1,2,\cdots,n$. If we let $\Sigma'(p)=M$, we have by (1.4), that $a(M)=e$. Let $b$ be any element of $L$ and let $c$ be in $\Sigma(b)$. Then $c\leqq e=a(M)$ $=\sup_{m\in M}\{a:m\}=\sup_{m\in M}\{\sup[N_m]\}\leqq\sup\left[\bigvee_{m\in M}N_m\right]$, where $N_m$ is the set of the compact elements $u$'s such that $\varphi(m)\varphi(u)\leqq a$. As is seen from the proof of (1.3), we can find a finite number of elements $x_i$ of $\bigvee_{m\in M}N_m$ (suppose $x_i$ in $N_{m_i}$) such that $c\leqq\bigcup_{i=1}^n x_i$ and there exists $m^*$ in $M^*$ (kernel of $M$) such that $\varphi(m^*)\leqq\varphi(m_i)$. Then we have $\varphi(c)\leqq\bigcup_{i=1}^n\varphi(x_i)$, and then $\varphi(m^*)\varphi(c)\leqq\bigcup_{i=1}^n\varphi(m^*)\varphi(x_i)\leqq\bigcup_{i=1}^n\varphi(m_i)\varphi(x_i)\leqq a\leqq p$. Since $p$ is $\varphi$-prime and $m^*\not\leqq p$, we have $c\leqq p$ for all $c$ in $\Sigma(b)$. This means that $\Sigma(b)$ is contained in $\Sigma(p)$, and then $b\leqq p$, for all $b$ in $L$, a contradiction.

As is easily seen from [3, Theorem 4.4] that if an element $a$ of $L$ has $\varphi$-primary decomposition, and let $a=q_1\cap q_2\cap\cdots\cap q_n$ be its normal decomposition, then the number of $\varphi$-primary components and the $\varphi$-radicals of $\varphi$-primary components depend only on $a$ and not on the particular normal decomposition considered. Then we have the following definition:

**Definition 2.2.** A subset $\{r(q_1), r(q_2), \cdots, r(q_m)\}$ of the radicals is called an *isolated set* of $a$, if for $m+1 \leqq j \leqq n, r(q_j) \not\leqq r(q_i)$ for $1 \leqq i \leqq m$.

**Proposition 2.3.** *Suppose that an element $a$ of $L$ has an $\varphi$-primary decomposition. Let $a = q_1 \cap q_2 \cap \cdots \cap q_n$ be its normal decomposition, and let $r(q_i) = \bigcap_k p_{ik}$ be the expression of $r(q_i)$ as the meet of all the quasi-minimal $\varphi$-prime elements belonging to $q_i$. Then the following three conditions are equivalent*:

(1) *The set $\{r(q_1), r(q_2), \cdots, r(q_m)\}$ is an isolated set of $a$.*

(2) *For each $q_i, 1 \leqq i \leqq m$, there exists at least one quasi-minimal $\varphi$-prime $p_{ik_i} = p_i^*$ such that $p_i^* \not\geqq p_{jk}$ for all $j, m+1 \leqq j \leqq n$ for all $k$.*

(3) *For each $r(q_i), 1 \leqq i \leqq m, r(q_i) \not\geqq q_{m+1} \cap q_{m+2} \cap \cdots \cap q_n$.*

**Proof.** We shall prove this proposition by the way that (1) implies (2), (2) implies (3) and (3) implies (1).

Suppose that $\{r(q_1), r(q_2), \cdots, r(q_m)\}$ is an isolated set of $a$. We have $r(q_j) \not\leqq r(q_i)$ for $1 \leqq i \leqq m, m+1 \leqq j \leqq n$ (by (2.2)). Assume that for every quasi-minimal $\varphi$-prime $p_{ik}$ of $r(q_i), p_{jk} \leqq p_{ik}$ for all $j, m+1 \leqq j \leqq n$ and for all $k$. This implies that $r(q_j) = \bigcap_k p_{jk} \leqq \bigcap_k p_{ik} = r(q_i)$ for all $j, m+1 \leqq j \leqq n$. It leads to a contradiction. Then there exists $p_{ik_i} = p_i^*$ such that $p_i^* \not\geqq p_{jk}$ for all $j, m+1 \leqq j \leqq n$ and for all $k$.

Secondly, we suppose that (2) is true. Since any $\varphi$-prime element greater than an element of $L$ must be greater than a quasi-minimal $\varphi$-prime element belonging to it [2], we have by $p_{jk} \not\leqq p_i^*$ that $q_j \not\leqq p_i^*$ for $1 \leqq i \leqq m$ and for all $j, m+1 \leqq j \leqq n$. Therefore we conclude that $\bigcap_j q_j \not\leqq p_i^*$ (because if not so, then we have $\bigcap_j q_j \leqq p_i^*$, by (2.1), we have $q_j \leqq p_i^*$ for some $j, m+1 \leqq j \leqq n$, a contradiction). Then $\bigcap_j q_j \not\leqq r(q_i)$ for $1 \leqq i \leqq m$ and $j = m+1, \cdots, n$.

Finally, we suppose $r(q_i) \not\geqq q_{m+1} \cap q_{m+2} \cap \cdots \cap q_n$, for $1 \leqq i \leqq m$. It follows that $r(q_i) \not\geqq q_j$, for all $j, m+1 \leqq j \leqq n$ and for $1 \leqq i \leqq m$. This implies that for all $i, 1 \leqq i \leqq m, r(q_i) \not\geqq r(q_j), m+1 \leqq j \leqq n$. Then $\{r(q_1), r(q_2), \cdots, r(q_m)\}$ is an isolated set of $a$.

Now, we come to the second uniqueness theorem for normal decomposition [3]. The proof of this theorem is just similar to [1, Theorem 8]. So, we will omit it here.

**Theorem 2.4.** *Suppose that an element $a$ has $\varphi$-primary decomposition, and let $a = q_1 \cap q_2 \cap \cdots \cap q_n$ be its normal decomposition. If $\{r(q_1), r(q_2), \cdots, r(q_m)\}$ is an isolated set of $a$, then $q_1 \cap q_2 \cap \cdots \cap q_m$ depends only on $r(q_1), r(q_2), \cdots, r(q_m)$ and not on the particular normal decomposition of $a$.*

**Remark.** By the second condition of (2.3), we have $p_{jk} \not\leq p_i^*$ for all $j, m+1 \leqq j \leqq n$ and for all $k$, where $p_i^* = p_{ik_i}$ and $1 \leqq i \leqq m$. Since any $\varphi$-prime element greater than an element of $L$ must be greater than a quasi-minimal $\varphi$-prime element belonging to it, we have $q_j \not\leq p_i^*$ for $1 \leqq i \leqq m$ and for all $j = m+1, \cdots, n$. (1.4) shows that for $1 \leqq i \leqq m$, each $a_{M_i}$ can be expressed as the meet of $q_1, q_2, \cdots, q_m$ and one of which is certainly $q_i$ (where $M_i = \Sigma'(p_i^*)$). Then we have

$$q_1 \cap q_2 \cap \cdots \cap q_m = q_{M_1} \cap q_{M_2} \cap \cdots \cap q_{M_m}.$$

Since each minimal element of the set $\{r(q_1), r(q_2), \cdots, r(q_n)\}$ forms on its own an isolated set of $a$, we have by (2.4) the following result which is analogous to that in [1].

**Corollary 2.5.** *Let $r(q_0)$ be a minimal element in the set $\{r(q_1), r(q_2), \cdots, r(q_n)\}$ of the radicals of the $\varphi$-primary components of $a$. Then the $\varphi$-primary component corresponding to $r(q_0)$ is the same for all normal decompositions of $a$.*

## References

[1] Y. Kurata and S. Kurata: A generalization of prime ideals in rings. Proc. Japan Acad., **45**, 75–78 (1969).

[2] K. Murata and Derbiau F. Hsu: Generalized prime elements in a compactly generated $l$-semigroup. I. Proc. Japan Acad., **49**, 134–139 (1973).

[3] ——: Generalized prime elements in a compactly generated $l$-semigroup. II. Proc. Japan Acad., **49**, 310–313 (1973).

[4] K. Murata, Y. Kurata, and H. Marubayashi: A generalization of prime ideals in rings. Osaka J. Math., **6**, 291–301 (1969).