135. *Direct Sum of Strongly Regular Rings and Zero Rings*

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1. Introduction. In [5] F. Szász investigated a class of rings, called P_1-rings, which coincides with the class of strongly regular rings in the absence of nilpotent elements. He showed that any P_1-ring is a subdirect sum of some zero rings of additive rank one and some division rings. In this paper, we shall give several characterizations of P_1-rings, in particular, it will be shown that any P_1-ring is a direct sum of a strongly regular ring and a zero ring. We also explore other generalizations of strongly regular rings and apply them to some commutatively theorems.

2. P_1-rings. Definition 1. A ring R is called a P_1-ring if $aR = aRa$ for each a in R.

We summarize here some of the results in [5] about P_1-rings.

Theorem 0. *Let R be a P_1-ring. Then*

(i) *$aR = aRa^n$ for any positive integer n and $NR = 0$ where N denotes the set of nilpotent elements of R.*

(ii) *R is strongly regular if and only if R has no nonzero nilpotent elements.*

Now we give a characterization of P_1-rings, but first a lemma is needed.

Lemma 1. *Let R be a P_1-ring. Then $ab = 0$ implies $ba = 0$ for any a, b in R.*

Proof. Suppose $ab = 0$. Then $baba = 0$ implies that ba is in N and from (i) of Theorem 0, $baR = 0$. R is P_1 implies that $ba = brb$ for some r in R. Hence $bar = brbr = 0$. Thus br is in N and $brR = 0$. Consequently $ba = brb = 0$.

Theorem 1. *A ring R is a P_1-ring if and only if*

(i) *$N \subseteq C$, where C denotes the center of R,*

(ii) *$E \subseteq C$, where E denotes the set of idempotents,*

(iii) *$NR = 0$,*

(iv) *R/N is strongly regular.*

Proof. Suppose R is a P_1-ring. If x is in N, then $xR = 0$. By Lemma 1, $Rx = 0$ and hence $N \subseteq C$. Now let $e = e^2$ be in R. Then for any x in R, $e(ex - x) = 0$ implies that $(ex - x)e = 0$ and $exe = xe$. Sim-

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ilarly, $exe=ex$. This proves (ii). The fact that $NR=0$ and R/N is strongly regular follows from (i) and (ii) of Theorem 0 respectively.

Conversely, we need to show that $aR=aRa$ for any a in R. Clearly $aRa\subseteq aR$. Since R/N is strongly regular, $(a+N)=(a+N)^2(b+N)$ for some b in R. But a strongly regular ring is a P_1-ring, so we have that $(a+N)(ab+N)=(a+N)(w+N)(a+N)$ for some w in R. Hence $(a-awa)$ is in N. From (i) and (iii), $w(a-awa)=0$. Thus wa is an idempotent and is in C. Now let r be in R. Then $(a-awa)r=0$ implies that $ar=awar=arwa$. Hence ar is in aRa and it follows that $aR=aRa$. Thus R is a P_1-ring.

The next result gives a complete structure of P_1-rings.

Theorem 2. *A ring R is P_1 if and only if it is a direct sum of a strongly regular ring and a zero ring.*

Proof. The if part is trivial. Suppose R is a P_1-ring. We wish to show that $R=R^2\oplus N$. Suppose $\sum x_iy_i$ is in $R^2\cap N$. Since R/N is strongly regular (hence regular), there is $(z+N)=(z+N)^2$ in R/N such that $(x_i+N)(z+N)=(x_i+N)$ and $x_i=x_iz+n_i, n_i\in N$ for each i. Since $NR=0, z^2$ is idempotent and $x_iz=x_iz^2$. Thus $\sum x_iy_i=\sum(x_iz+n_i)y_i$ $=\sum x_izy_i=\sum x_iz^2y_i=\sum(x_iy_i)z^2=0$. Thus $N\cap R^2=0$. Also for each x in R, there is a y in R such that $(x-x^2y)$ is in N. Hence $R=R^2+N$. From (iii) of Theorem 1, $N^2=0$. It remains to show that R^2 is strongly regular. Since $aR^2=aRaR=aRaRa=aR^2a, R^2$ is a P_1-ring with no nilpotent elements and thus is strongly regular.

Remark. From Theorem 2 it follows that any P_1-ring R with d.c.c. on right ideals is a direct sum of division rings and a zero ring. In particular, if R is finite, R is a commutative ring.

Theorem 3. *For an arbitrary ring R the following are equivalent:*

(1) *R is a direct sum of a strongly regular ring and a zero ring.*

(2) *R is a P_1-ring.*

(3) *$aR=Ra^2$ for any a in R.*

(4) *$aR\subseteq Ra^2$ for any a in R and any idempotent of R is central.*

Proof. (1) implies (2) follows from Theorem 2. (2) implies (3): Since $aR=aRa^2$ for each a in $R, aR\subseteq Ra^2$. Let r be in R. Since R/N is strongly regular, there is an x in R such that (a^2-a^4x) is in N. By (ii) of Theorem 1 and Lemma 1, $(a^2-a^4x)r=0=a^2(r-a^2xr)=(r-a^2xr)a^2$. Hence ra^2 is in aR and $aR=Ra^2$.

(3) implies (4): For $e^2=e, eR=Re$. Hence if a is in R, $ea=xe$ implies $eae=xe=ea$. Similarly, $eae=ae$. Thus e is central.

(4) implies (1): Since $aR^2\subseteq Ra^2R$, it follows that for any $n\geqslant 1$, $Ra^nR\subseteq Ra^{2n}\subseteq Ra^{2n-1}R$, and so $aR^2\subseteq Ra^2R\subseteq Ra^3R\subseteq Ra^5R\cdots$. This shows that $aR^2=0$ for any nilpotent a. Thus the set of nilpotent elements of R forms an ideal N of R. Now $a^2=ba^2$ for some b. Hence $(a-ba)a$

$=0$ and $(a-ab)a=a(a-ba)$. A quick calculation shows that $(a-ab)^2$ is in N and hence $(a-ab)$ is in N. Since $aR\subseteq Ra^2$, $ab=xa^2$ for some x in R. We see that $(a-xa^2)$ is in N and hence R/N is strongly regular. Since $aR^3\subseteq Ra^2R^2\subseteq R^2a^4R\subseteq R^3a^8\subseteq R^3a^2$, it follows that R^3 satisfies the assumption (4). Suppose $\sum x_iy_iz_i$ is a nilpotent element in R^3. Since R/N is regular, $x_i=x_ie\ (\text{mod}\ N)$ for some $e^2=e$. Hence $(x_i-x_ie)R^2=0$ and $x_iy_iz_i=x_iey_iz_i$. Since any idempotent is central, we see that $\sum x_iy_iz_i=\sum (x_iy_iz_i)e^2=0$. Thus R^3 has no nonzero nilpotent elements, and is strongly regular from the argument above. It remains to show that $R=R^3+N$ and that N is a zero ring. The fact that R/N is strongly regular implies that for each a in R, there is a b in R such that $(a-a^2b)$ is in N. Hence $a\in R^3+N$ and $R=R^3\oplus N$. Since N is a direct summand of R, N satisfies (4) and $aN\subseteq Na^2\subseteq NR^2=0$ for any a in N. Thus N is a zero ring.

3. Generalizations. In this section we consider other classes of rings having the property that $ab=0$ implies $ba=0$. By Lemma 1, a P_1-ring has this property. Now we adopt the following definition.

Definition 2. Let R be a ring. Then R is called

(a) a P_2-ring if for each x, y in R, there is a positive integer $n=n(x,y)>1$ and an element $z=z(x,y)$ in the center of R, such that $(xy-yx)=(xy-yx)^nz$.

(b) a P_3-ring if every homomorphic image R' of R has the property that $ab=0$ implies $ba=0$ for each a, b in R'.

(c) a P_4-ring if for each x in R, $A(x)=\{y\in R: xy=0\}$ is an ideal.

Lemma 2. *Let R be a P_i $(i=1,2)$ ring. Then R is a P_3-ring and any P_3-ring is a P_4-ring.*

Proof. A P_1-ring is a P_3-ring by Lemma 1. Any P_2-ring in a P_3-ring is given in the proof of Theorem 1 in [1] with the obvious modification. Clearly a P_3-ring is a P_4-ring.

The class of P_2-rings was studied in [3] and [4]. In fact it was shown in [3] that a P_2-ring is a subdirect sum of commutative rings and division rings. We studied P_4-rings with d.c.c. in [2]. Now we consider P_3-rings and apply it to a commutativity theorem of Herstein.

Theorem 4. *Let R be a P_3-ring. Then R is a subdirect sum of subdirectly irreducible rings R_i where each R_i is one of the following types:*

(A) *R_i is a zero ring,*

(B) *R_i has no zero divisors,*

(C) *Any nonzero idempotent in R_i is the identity.*

Proof. Since any ring R is a subdirect sum of subdirectly irreducible rings R_i, it follows that $ab=0$ implies $ba=0$ for each a, b in R_i.

Case 1. R_i has the zero multiplication. This is type (A).

Case 2. R_i has no proper ideals. Since for each x in R_i, $A(x)$ is an ideal, it follows that R_i has no zero divisors. This is type (B).

Case 3. R_i has proper ideals. If $0\neq e=e^2$ is in R_i, then e is the identity. For if not, then eR and $A(e)$ are ideals and $eR\cap A(e)=0$. Since R_i is subdirectly irreducible, R_i is of type (C).

From Theorem 4 one can obtain the following corollaries whose proofs we shall omit.

Corollary 1 ([3]). *Any P_2-ring is a subdirect sum of commutative rings and division rings.*

Corollary 2 (Herstein). *R is a commutative ring if and only if for each x, y in R, there is an integer $n=n(x,y)>1$, such that $(xy-yx)=(xy-yx)^n$.*

Corollary 3 ([4, Theorem 5]). *If R is a P_2-ring with $n=2$, then R is commutative.*

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