134. *On Submodules over an Asano Order of a Ring*[*)]

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1. Let R be a ring with unity quantity, and let $\mathfrak{o}$ be a regular maximal order of R. The term *ideal* means a non-zero fractional two-sided $\mathfrak{o}$-ideal in R. We shall use small German letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ with or without suffices to denote ideals in R. The inverse of an ideal $\mathfrak{a}$ will be denoted by $\mathfrak{a}^{-1}$, and $\mathfrak{a}^*$ will denote $\mathfrak{a}^{-1-1}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *quasi-equal* if $\mathfrak{a}^{-1}=\mathfrak{b}^{-1}$; in symbol: $\mathfrak{a}\sim\mathfrak{b}$. The term *submodule* means a two-sided $\mathfrak{o}$-submodule which contains at least one regular element of R. A submodule M is said to be *closed* if whenever $\mathfrak{a}\subseteq M$ implies $\mathfrak{a}^*\subseteq M$. It is then clear that every submodule is closed when the arithmetic holds for $\mathfrak{o}$ (cf. [1, § 2]). For any two closed submodules M_1 and M_2 we define a product $M_1\circ M_2$ to be the set-theoretical union of all ideals $(\sum_{i=1}^n \mathfrak{a}_i\mathfrak{b}_i)^*$ where $\mathfrak{a}_i\subseteq M_1$ and $\mathfrak{b}_i\subseteq M_2$ $(i=1,\cdots,n)$. Now the set G of all ideals $\mathfrak{a}$ such that $\mathfrak{a}=\mathfrak{a}^*$ forms a *commutative* group under the multiplication "$\circ$" defined by $\mathfrak{a}\circ\mathfrak{b}=(\mathfrak{a}\mathfrak{b})^*=(\mathfrak{a}^*\mathfrak{b}^*)^*$; because G is a (conditionally) complete l-group under the above multiplication and the inclusion (cf. p. 91 in [5]). Hence $M_1\circ M_2=M_2\circ M_1$, and if the ascending chain condition in the sense of quasi-equality holds for integral ideals, the set $\mathfrak{M}$ of all closed submodules forms a commutative l-semigroup under the above multiplication and the set-inclusion (cf. Lemmas 5.1 and 5.2 in [2]).

Let $\mathfrak{P}$ be the set of all prime ideals which are not quasi-equal to $\mathfrak{o}$, let $|\mathfrak{P}|$ be the cardinal number of $\mathfrak{P}$, and let $\mathbf{Z}_{-\infty}$ be the set-theoretical union of the rational integers $\mathbf{Z}$ and $-\infty$. Then the complete direct sum $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$ ($|\mathfrak{P}|$-copies) of $\mathbf{Z}_{-\infty}$ is an l-semigroup under the addition $[m_\mathfrak{p}]+[n_\mathfrak{p}]=[m_\mathfrak{p}+n_\mathfrak{p}]$ and the partial order $[m_\mathfrak{p}]\succ[n_\mathfrak{p}]\Leftrightarrow m_\mathfrak{p}\leq n_\mathfrak{p}$ for all $\mathfrak{p}\in\mathfrak{P}$, where $m_\mathfrak{p}, n_\mathfrak{p}\in\mathbf{Z}_{-\infty}$. Let $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$ be the set of all vectors $[m_\mathfrak{p}]$ such that $m_\mathfrak{p}\leq 0$ for almost all $\mathfrak{p}\in\mathfrak{P}$. Then it forms an l-subsemigroup of $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$.

The aim of the present note is to prove the following

Theorem. *If the ascending chain condition in the sense of quasi-equality* (cf. p. 109 in [1]) *holds for integral ideals, the l-semigroup $\mathfrak{M}$ of all non-zero closed submodules is isomorphic to $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$ as an l-semigroup. If in particular the arithmetic holds for $\mathfrak{o}$, the l-semigroup $\mathfrak{M}$ of all submodules* (*containing regular elements*) *is isomorphic to $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-}$ as an l-semigroup, and every submodule $M\in\mathfrak{M}$ is written as follows:*

*) Dedicated to professor Kiiti Morita on his 60th birthday.

$$M=\prod_{\mathfrak{p}\in P_+}\mathfrak{p}^{\nu(\mathfrak{p})}\Big(\sum_{\mathfrak{p}\in P_-}\mathfrak{p}^{\nu(\mathfrak{p})}\Big)\mathfrak{o}_P \qquad (*)$$

where $\nu(\mathfrak{p})=\nu_M(\mathfrak{p})$ *is the* $\mathfrak{p}$*-coordinate of the vector in* $\bigoplus_{\mathfrak{P}}^{*}\mathbf{Z}_{-\infty}$ *which corresponds to* M *by the above isomorphism,* $P_+=P_+(M)$ *is the prime ideals with* $\nu_M(\mathfrak{p})>0$, $P_-=P_-(M)$ *is the prime ideals with* $-\infty<\nu_M(\mathfrak{p})<0$, $\mathfrak{o}_P$ *is the* P*-component of* $\mathfrak{o}$ (cf. [1, § 3]) *for the set* $P=P_0(M)\cup P_+(M)\cup P_-(M)$ *the prime ideals with* $\nu_M(\mathfrak{p})=0$, *and* Σ *denotes the restricted direct sum.*

$P_+(M)$ is a finite set for each submodule M, but both $P_-(M)$ and $P_0(M)$ are not necessarily finite.

The first half of Theorem is a generalization of [3, Theorem 1] in the case of Dedekind domains (cf. [4, § 2]) to a non-commutative case.

2. Proof of Theorem. Let $\mathfrak{a}$ be an ideal, and let $\mathfrak{a}\sim\mathfrak{a}^*\doteq\mathring{\Pi}\mathfrak{p}^{\alpha}$, $\alpha\in\mathbf{Z}$, be the factorization of $\mathfrak{a}^*$ into prime ideals $\mathfrak{p}$'s with $\mathfrak{p}\not\sim\mathfrak{o}$, where $\mathring{\Pi}_i\mathfrak{a}_i$ means $(\Pi_i\mathfrak{a}_i)^*$ (cf. p. 13 in [2]). In the following we use $\nu(\mathfrak{p};\mathfrak{a})$ to denote α, the $\mathfrak{p}$-exponent of $\mathfrak{a}^*$. Then we have

(1) $\nu(\mathfrak{p};\mathfrak{a})=0$ for almost all $\mathfrak{p}\in\mathfrak{P}$.

(2) $\nu(\mathfrak{p};\mathfrak{a})=\nu(\mathfrak{p};\mathfrak{a}^*)$.

(3) $\nu(\mathfrak{p};\mathfrak{a}+\mathfrak{b})=\mathrm{Min}\,\{\nu(\mathfrak{p};\mathfrak{a}),\ \nu(\mathfrak{p};\mathfrak{b})\}$.

(4) $\nu(\mathfrak{p};\mathfrak{a}\mathfrak{b})=\nu(\mathfrak{p};\mathfrak{a})+\nu(\mathfrak{p};\mathfrak{b})$.

(5) $\mathfrak{a}\subseteq\mathfrak{b}$ implies $\nu(\mathfrak{p};\mathfrak{a})\geq\nu(\mathfrak{p};\mathfrak{b})$.

(6) If $\nu(\mathfrak{p};\mathfrak{a})\geq\nu(\mathfrak{p};\mathfrak{b})$ for all $\mathfrak{p}\in\mathfrak{P}$, then $\mathfrak{a}\subseteq\mathfrak{b}^*$.

(7) If $\nu(\mathfrak{p};\mathfrak{a})=\nu(\mathfrak{p};\mathfrak{b})$ for all $\mathfrak{p}\in\mathfrak{P}$, then $\mathfrak{a}\sim\mathfrak{b}$.

Ad (3): It follows from $(\mathfrak{a}+\mathfrak{b})^*=(\mathfrak{a}^*+\mathfrak{b}^*)^*$. Ad (4): It follows from $(\mathfrak{a}\mathfrak{b})^*=(\mathfrak{a}^*\mathfrak{b}^*)^*$ (cf. p. 13 in [2]). (5) is immediate from (3). The other properties are evident.

The initial stage in our proof will be a generalization of $\nu(\mathfrak{p};\ )$ for submodules. For any $M\in\mathfrak{M}$ we define

$$\nu(\mathfrak{p};M)=\inf\{\nu(\mathfrak{p};\mathfrak{a})\mid\mathfrak{a}\subseteq M\}.$$

Then, fixing M and running $\mathfrak{p}$ through $\mathfrak{P}$, $\nu(\mathfrak{p};M)$ is considered as a map from $\mathfrak{P}$ into $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$. In this state it is convenient to use $\nu_M(\mathfrak{p})$ or ν_M instead of $\nu(\mathfrak{p};M)$. For any fixed ideal $\mathfrak{a}_0$ in M we have $\nu_M(\mathfrak{p})\leq\nu(\mathfrak{p};\mathfrak{a}_0)$. Hence $\nu_M(\mathfrak{p})\leq 0$ for almost all $\mathfrak{p}\in\mathfrak{P}$.

Let σ be a map from $\mathfrak{P}$ into $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$ such that $\sigma(\mathfrak{p})\leq 0$ for almost all $\mathfrak{p}\in\mathfrak{P}$, and let $M\langle\sigma\rangle$ be the set-theoretical union of all ideals $\mathfrak{a}$ such that $\nu(\mathfrak{p};\mathfrak{a})\geq\sigma(\mathfrak{p})$ for all $\mathfrak{p}\in\mathfrak{P}$. Then $M\langle\sigma\rangle$ is a closed submodule in our sense. For, we let $\mathfrak{b}$ be an ideal contained in $M\langle\sigma\rangle$. Then by the ascending chain condition in the sense of quasi-equality and by the regularity of $\mathfrak{o}$, we can choose a finite number of elements $b_1,\cdots,b_n$ in $\mathfrak{b}$ such that at least one of the b_i is regular and $\mathfrak{b}^*=(b_1,\cdots,b_n)^*$. Taking $\mathfrak{a}_i$ such that $\mathfrak{a}_i\ni b_i$, $\mathfrak{a}_i\subseteq M\langle\sigma\rangle$, we have $\mathfrak{b}^*=(b_1,\cdots,b_n)^*\subseteq(\sum_{i=1}^n\mathfrak{a}_i^*)^*=(\sum_{i=1}^n\mathfrak{a}_i)^*$. Hence $\nu(\mathfrak{p};\mathfrak{b}^*)\geq\nu(\mathfrak{p};(\sum_{i=1}^n\mathfrak{a}_i)^*)=\nu(\mathfrak{p};\sum_{i=1}^n\mathfrak{a}_i)=\mathrm{Min}\,\{\nu(\mathfrak{p};\mathfrak{a}_i)\}\geq\sigma(\mathfrak{p})$. Thus we get $\mathfrak{b}^*\subseteq M\langle\sigma\rangle$.

We note here that for each ideal $\mathfrak{a}\subseteq M\langle\nu_M\rangle$ there exists an ideal $\mathfrak{c}$ such that $\nu(\mathfrak{p};\mathfrak{c})\leq\nu(\mathfrak{p};\mathfrak{a})$, $\mathfrak{c}\subseteq M$. For, if there is no such ideal we have $\nu(\mathfrak{p};\mathfrak{c})>\nu(\mathfrak{p};\mathfrak{a})$ for all (non-zero) ideal $\mathfrak{c}\subseteq M$. Since $\nu(\mathfrak{p};\mathfrak{a})\neq-\infty$, the set of all $\nu(\mathfrak{p};\mathfrak{c})$, $\mathfrak{c}\subseteq M$, has a lower bound. Hence there exists an integer n_0 such that $\nu(\mathfrak{p};M)=n_0=\nu(\mathfrak{p};\mathfrak{c}_0)$ for a suitable $\mathfrak{c}_0\subseteq M$. By the assumption we have $n_0=\nu(\mathfrak{p};\mathfrak{c}_0)>\nu(\mathfrak{p};\mathfrak{a})$. However $\mathfrak{a}\subseteq M\langle\nu_M\rangle$ implies $\nu(\mathfrak{p};\mathfrak{a})\geq\nu_M(\mathfrak{p})=n_0$, which is a contradiction.

Now we prove $M\langle\nu_M\rangle=M$. $M\subseteq M\langle\nu_M\rangle$ is evident. Conversely, let $\mathfrak{a}$ be an arbitrary (non-zero) ideal in $M\langle\nu_M\rangle$, and let $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$ be the all prime ideals $\mathfrak{p}$ such that $\nu(\mathfrak{p};\mathfrak{a})\neq0$, $\mathfrak{p}\in\mathfrak{P}$. Then we can choose a suitable ideal $\mathfrak{c}_1$ such that $\nu(\mathfrak{p}_1;\mathfrak{c}_1^*)\leq\nu(\mathfrak{p}_1;\mathfrak{a})$, $\mathfrak{c}_1\subseteq M$. Next we let $\mathfrak{p}_{m+1},\cdots,\mathfrak{p}_n$ be all prime ideals $\mathfrak{p}$, if there exists, such that $\nu(\mathfrak{p};\mathfrak{c}_1)>0$ and $\mathfrak{p}$ does not appear among $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$. Then we can take suitable ideals $\mathfrak{c}_i$ such that $\nu(\mathfrak{p}_i;\mathfrak{c}_i)\leq\nu(\mathfrak{p}_i;\mathfrak{a})$, $\mathfrak{c}_i\subseteq M$ $(i=2,\cdots,n)$. Then clearly $\mathfrak{c}=\mathfrak{c}_1+\mathfrak{c}_2+\cdots+\mathfrak{c}_n\subseteq M$, and $\mathfrak{c}^*\subseteq M$. For any $\mathfrak{p}_j$ $(j=1,\cdots,n)$, we have $\nu(\mathfrak{p}_j;\mathfrak{c})\leq\nu(\mathfrak{p}_j;\mathfrak{c}_j)\leq\nu(\mathfrak{p}_j;\mathfrak{a})$, and for any $\mathfrak{p}\in\mathfrak{P}$ different from $\mathfrak{p}_j$ $(j=1,\cdots,n)$, we have $\nu(\mathfrak{p};\mathfrak{c})\leq\nu(\mathfrak{p};\mathfrak{c}_1)\leq0=\nu(\mathfrak{p};\mathfrak{a})$. Thus we obtain $\mathfrak{a}\subseteq\mathfrak{c}^*$, $\mathfrak{a}\subseteq M$ as desired.

Next we prove $\nu_{M\langle\sigma\rangle}=\sigma$. Let $\mathfrak{p}_1,\cdots,\mathfrak{p}_n$ be the set of all the prime ideals $\mathfrak{p}$ such that $\sigma(\mathfrak{p})>0$, $\mathfrak{p}\in\mathfrak{P}$. We form $\mathfrak{c}=\mathfrak{p}_1^{\sigma(\mathfrak{p}_1)}\circ\cdots\circ\mathfrak{p}_n^{\sigma(\mathfrak{p}_n)}$. Then evidently $\mathfrak{c}^*=\mathfrak{c}$ and $\nu(\mathfrak{p}_i;\mathfrak{c})=\sigma(\mathfrak{p}_i)$ for $i=1,\cdots,n$. If $\mathfrak{p}\neq\mathfrak{p}_i$ $(i=1,\cdots,n)$, $\mathfrak{p}\in\mathfrak{P}$, then $\nu(\mathfrak{p};\mathfrak{c})=0\geq\sigma(\mathfrak{p})$. Hence $\mathfrak{c}\subseteq M\langle\sigma\rangle$, and hence $\nu(\mathfrak{p}_i;M\langle\sigma\rangle)\leq\nu(\mathfrak{p}_i;\mathfrak{c})=\sigma(\mathfrak{p}_i)$ for $i=1,\cdots,n$. If $\mathfrak{p}'\neq\mathfrak{p}_i$ $(i=1,\cdots,n)$, $\mathfrak{p}'\in\mathfrak{P}$, then putting $\mathfrak{a}=(\mathfrak{c}\mathfrak{p}'^{\sigma(\mathfrak{p}')})^*$, we have $\nu(\mathfrak{p}_i;\mathfrak{a})=\sigma(\mathfrak{p}_i)$ and $\nu(\mathfrak{p}';\mathfrak{a})=\sigma(\mathfrak{p}')$. For any $\mathfrak{p}''$ such that $\mathfrak{p}''\neq\mathfrak{p}_i$ $(i=1,\cdots,n)$, $\mathfrak{p}''\neq\mathfrak{p}'$, $\mathfrak{p}''\in\mathfrak{P}$, we have $\nu(\mathfrak{p}'';\mathfrak{a})=0\geq\sigma(\mathfrak{p}'')$. Hence $\mathfrak{a}\subseteq M\langle\sigma\rangle$, and hence $\nu(\mathfrak{p}';M\langle\sigma\rangle)\leq\nu(\mathfrak{p}';\mathfrak{a})=\sigma(\mathfrak{p}')$ for an arbitrary $\mathfrak{p}'\neq\mathfrak{p}_i$ $(i=1,\cdots,n)$, $\mathfrak{p}'\in\mathfrak{P}$. Above all we get $\nu(\mathfrak{p};M\langle\sigma\rangle)\leq\sigma(\mathfrak{p})$ for all $\mathfrak{p}\in\mathfrak{P}$. Thus we have $\nu_{M\langle\sigma\rangle}\leq\sigma$. $\nu_{M\langle\sigma\rangle}\geq\sigma$ is evident by the definition of $\nu_{M\langle\sigma\rangle}$. Therefore we obtain $\nu_{M\langle\sigma\rangle}=\sigma$ as desired.

By the above argument we have

$$M\mapsto\nu_M\mapsto M\langle\nu_M\rangle=M,\qquad \sigma\mapsto M\langle\sigma\rangle\mapsto\nu_{M\langle\sigma\rangle}=\sigma.$$

Accordingly the map $M\mapsto\nu_M$ gives a bijection from $\mathfrak{M}$ to the set of all σ. Now it is clear that the set of all vectors $[\sigma(\mathfrak{p})]=\{\sigma(\mathfrak{p})\mid\mathfrak{p}\in\mathfrak{P}\}$ coincides with $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$. We shall show the map f:

$$M\mapsto f(M)=[\nu_M(\mathfrak{p})]$$

gives an l-semigroup-isomorphism from $\mathfrak{M}$ to $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$. For, let M_1, $M_2\in\mathfrak{M}$, and take an arbitrary (non-zero) ideal $\mathfrak{c}$ contained in $M_1\circ M_2$. Then by using the ascending chain condition in the sense of quasi-equality for integral ideals we can take an ideal $(\sum_{i=1}^n\mathfrak{a}_i\mathfrak{b}_i)^*$ which contains $\mathfrak{c}$. In fact by the ascending chain condition in the sense of quasi-equality $\mathfrak{c}^*$ is generated by a finite number of elements $x_1,\cdots,x_m$ in $\mathfrak{c}$ (some of x_k is regular), i.e., $\mathfrak{c}^*=(x_1,\cdots,x_m)^*$, $x_k\in\mathfrak{c}$. Then by the definition of $M_1\circ M_2$, we can take $(\sum_{i=1}^{n(k)} \mathfrak{a}_i^{(k)}\mathfrak{b}_i^{(k)})^*$ which contains x_k $(k=1,\cdots,m)$ where $\mathfrak{a}_i^{(k)}\subseteq M_1$ and $\mathfrak{b}_i^{(k)}\subseteq M_2$. Hence $x_j \in \sum_{k=1}^{m}(\sum_{i=1}^{n(k)} \mathfrak{a}_i^{(k)}\mathfrak{b}_i^{(k)})^*$ $(j=1,\cdots,m)$, and $\mathfrak{c}\subseteq\mathfrak{c}^*=(x_1,\cdots,x_m)^*\subseteq(\sum_{k=1}^{m}(\sum_{i=1}^{n(k)} \mathfrak{a}_i^{(k)}\mathfrak{b}_i^{(k)})^*)^*$ $=(\sum_{k=1}^{m}\sum_{i=1}^{n(k)} \mathfrak{a}_i^{(k)}\mathfrak{b}_i^{(k)})^*\equiv(\sum_{i=1}^{n}\mathfrak{a}_i\mathfrak{b}_i)^*$. Then we have $\nu(\mathfrak{p}\,;\mathfrak{c})\geq\nu(\mathfrak{p}\,;(\sum_{i=1}^{n}\mathfrak{a}_i\mathfrak{b}_i)^*)$ $=\mathrm{Min}\,\{\nu(\mathfrak{p}\,;\mathfrak{a}_i)+\nu(\mathfrak{p}\,;\mathfrak{b}_i)\}\geq\inf\{\nu(\mathfrak{p}\,;\mathfrak{a}_i)\mid\mathfrak{a}_i\subseteq M_1\}+\inf\{\nu(\mathfrak{p}\,;\mathfrak{b}_i)\mid\mathfrak{b}_i\subseteq M_2\}$ $=\nu(\mathfrak{p}\,;M_1)+\nu(\mathfrak{p}\,;M_2)$. This implies $\nu(\mathfrak{p}\,;M_1\circ M_2)=\inf\{\nu(\mathfrak{p}\,;\mathfrak{c})\mid\mathfrak{c}^*\subseteq M_1\circ M_2\}$ $\geq\nu(\mathfrak{p}\,;M_1)+\nu(\mathfrak{p}\,;M_2)$. Since $\nu(\mathfrak{p}\,;\mathfrak{a})+\nu(\mathfrak{p}\,;\mathfrak{b})=\nu(\mathfrak{p}\,;\mathfrak{a}\mathfrak{b})\geq\nu(\mathfrak{p}\,;M_1\circ M_2)$ for any $\mathfrak{a}\subseteq M_1$ and $\mathfrak{b}\subseteq M_2$, we have $\nu(\mathfrak{p}\,;M_1)+\nu(\mathfrak{p}\,;\mathfrak{b})=\inf_{\mathfrak{a}\subseteq M_1}\{\nu(\mathfrak{p}\,;\mathfrak{a})+\nu(\mathfrak{p}\,;\mathfrak{b})\}$ $\geq\nu(\mathfrak{p}\,;M_1\circ M_2)$, $\nu(\mathfrak{p}\,;M_1)+\nu(\mathfrak{p}\,;M_2)\geq\nu(\mathfrak{p}\,;M_1\circ M_2)$. Hence the opposite inequality is true. It is evident that f is order-preserving. f is therefore an l-semigroup-isomorphism from $\mathfrak{M}$ to $\bigoplus_{\mathfrak{P}}^{*}\mathbf{Z}_{-\infty}$. If the arithmetic holds for $\mathfrak{o}$, then the l-semigroup $\mathfrak{M}$ of all submodules containing regular elements is isomorphic to $\bigoplus_{\mathfrak{P}}^{*}\mathbf{Z}_{-\infty}$ as an l-semigroup.

In order to prove the last part of the theorem we show that a submodule M is a subring containing $\mathfrak{o}$, if and only if the coordinates of the vector $f(M)=[\nu_M(\mathfrak{p})]$ consists only of 0 and $-\infty$; and in this case $M=\mathfrak{o}_P$ the P-component of $\mathfrak{o}$ where $P=P_0(M)$. We suppose that M is a subring which contains $\mathfrak{o}$ strictly. Since there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}^{-1}\subseteq M$, $\mathfrak{p}\in\mathfrak{P}$ (cf. Hilfssatz 6, p. 119 in [1]), we have $\mathfrak{p}^{-n}\subseteq M$ for all $n\in\mathbf{Z}^+$, the positive integers. Hence we obtain $\nu_M(\mathfrak{p})$ $=\inf\{\nu(\mathfrak{p}\,;\mathfrak{a})\mid\mathfrak{a}\subseteq M\}\leq\inf\{\nu(\mathfrak{p}\,;\mathfrak{p}^{-n})\mid n\in\mathbf{Z}^+\}=\inf\{-n\mid n\in\mathbf{Z}^+\}=-\infty$. If $\mathfrak{p}^{-1}$ is not contained in M, we can show $\nu_M(\mathfrak{p})=0$ as follows: Since $\mathfrak{o}\subsetneqq M$, M contains a pure fractional ideal. Let F be the set of the pure fractional ideals in M. Then evidently $\nu_M(\mathfrak{p})\leq\inf\{\nu(\mathfrak{p}\,;\mathfrak{b})\mid\mathfrak{b}\in F\}\equiv\alpha$. To prove the opposite inequality we take an arbitrary ideal $\mathfrak{a}$ in M. Then there exists a pure fractional ideal $\mathfrak{a}'$ such that $\mathfrak{a}\subseteq\mathfrak{a}'\subseteq M$ (e.g. $\mathfrak{a}'=\mathfrak{a}+\mathfrak{o}$). Then we have $\nu(\mathfrak{p}\,;\mathfrak{a})\geq\nu(\mathfrak{p}\,;\mathfrak{a}')\geq\alpha$. Hence we get $\nu_M(\mathfrak{p})$ $=\inf\{\nu(\mathfrak{p}\,;\mathfrak{b})\mid\mathfrak{b}\in F\}$. Suppose that there exists an ideal $\mathfrak{b}\in F$ such that $\mathfrak{p}^{-1}$ appears among the prime factors of $\mathfrak{b}$, $\mathfrak{b}=\mathfrak{p}^{-1}\cdot\mathfrak{b}'$, say. Then we have $\mathfrak{p}^{-1}\subseteq\mathfrak{b}\subseteq M$, a contradiction. Hence $\nu(\mathfrak{p}\,;\mathfrak{b})=0$ for all $\mathfrak{b}\in F$. We have therefore $\nu_M(\mathfrak{p})=0$ as desired. Conversely let M be a submodule such that the coordinates of $f(M)$ consists only of 0 and $-\infty$. An ideal $\mathfrak{a}$ is contained in M if and only if both $P_0(M)\subseteq P_0(\mathfrak{a})\cup P_+(\mathfrak{a})$ and $P_{-\infty}(M)\subseteq P_0(\mathfrak{a})\cup P_{\pm}(\mathfrak{a})$ hold, where $P_{-\infty}(M)=\{\mathfrak{p}\in\mathfrak{P}\mid\nu_M(\mathfrak{p})=-\infty\}$. In order to show that M is a subring of R it is sufficient to show that $\mathfrak{a}\mathfrak{b}$ $\subseteq M$ for any ideals $\mathfrak{a}$ and $\mathfrak{b}$ in M. Because, since $\mathfrak{o}$ is regular there is an ideal which is contained in M and contains an arbitrary fixed element of M. Take two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ in M. Then since $f(\mathfrak{a}\mathfrak{b})=f(\mathfrak{a})+f(\mathfrak{b})$ we can show $P_0(M)\subseteq P_0(\mathfrak{a}\mathfrak{b})\cup P_+(\mathfrak{a}\mathfrak{b})$ and $P_{-\infty}(M)$ $\subseteq P_0(\mathfrak{a}\mathfrak{b})\cup P_{\pm}(\mathfrak{a}\mathfrak{b})$. This means $\mathfrak{a}\mathfrak{b}\subseteq M$. $M=\mathfrak{o}_P$, $P=P_0(M)$, is easy to see. The representation $(*)$ is obtained by using the additive property of f. This completes the proof.

Remark. Let M be a submodule such that $|P_{-}(M)|$ is finite. Then $\sum \mathfrak{p}^{\nu(\mathfrak{p})}=(\prod \mathfrak{p}^{-\nu(\mathfrak{p})})^{-1}$, and M is the P-component of the ideal

$$\prod_{\mathfrak{p}\in P_+} \mathfrak{p}^{\nu(\mathfrak{p})} \prod_{\mathfrak{p}\in P_-} \mathfrak{p}^{\nu(\mathfrak{p})}.$$

Moreover $M=\mathfrak{a}_P$ (the P-component of an ideal $\mathfrak{a}$) if and only if

$$\mathfrak{a}=\prod_{\mathfrak{p}\in P_+} \mathfrak{p}^{\nu(\mathfrak{p})} \prod_{\mathfrak{p}\in P_-} \mathfrak{p}^{\nu(\mathfrak{p})} \prod_{\mathfrak{p}\in Q} \mathfrak{p}^{\rho},$$

where Q is a finite subset of $P_{-\infty}(M)$ and ρ is an integer. It is then obvious that a submodule M is a P-component of an ideal if and only if both $P_{-\infty}(\mathfrak{o}_P)=P_{-\infty}(M)$ and $|P_0(\mathfrak{o}_P)-P_0(M)|<\infty$ hold.

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