128. *Energy Inequalities and Finite Propagation Speed of the Cauchy Problem for Hyperbolic Equations with Constantly Multiple Characteristics*

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Let $P(t,x;D_t,D_x)$ be a differential polynomial defined in a domain $\Omega=[0,T]\times R^n$, $T>0$, of the form $P(t,x;D_t,D_x)=D_t^m+\sum_{j=0}^{m-1}\sum_{j+|\nu|\leqq m} a_{j,\nu}(t,x)D_t^jD_x^\nu$, $a_{j,\nu}\in\mathcal{B}(\Omega)$, with $D_t=-i\partial/\partial t$ and $D_x=(-i\partial/\partial x_1,\cdots,-i\partial/\partial x_n)$, and let us consider the Cauchy problem:

$$(1)\qquad \begin{cases} Pu=f & \text{in } (0,T)\times R^n \\ D_t^ju(0,x)=u_j, & j=0,1,\cdots,m-1 \end{cases}$$

for given $f\in C^\infty(\Omega)$ and $u_j\in C^\infty(R^n)$. It is well-known that the characteristic roots are real if the Cauchy problem is well-posed in C^∞ (cf. [3]). In the present note we study a sufficient conditions for the problem (1) to be well posed when charateristics are real and have constant multiplicity. Concerning this problem, S. Mizohata and Y. Ohya [4], [5] obtained a necessary and sufficient condition when the multiplicity is less than 2, and Y. Ohya [6] studied a sufficient condition when the multiplicity is less than 3. Recently, J. Chazarain [1] discusses the case of the arbitrary multiplicity by making use of the theory of Fourier integral operators. Our arguments seem to be different from his.

1. E. E. Levi's condition and the main theorem. Let the principal part P_m of P be written as $P_m(t,x;\tau,\xi)=\prod_{j=0}^{r}(\tau-\lambda_j(t,x;\xi))^{l_j}$, and assume that $\lambda_j(t,x;\xi)$, $1\leqq j\leqq r$, are real for $\xi\in R^n-\{0\}$ and $\inf_{(t,x)\in\Omega,|\xi|=1}|\lambda_j(t,x;\xi)-\lambda_k(t,x;\xi)|\geqq d>0$, $j\neq k$. Moreover, without loss of generality we may assume that $l_1=l_2\cdots=l_{r_1}>l_{r_1+1}=\cdots=l_{r_2}>\cdots>\cdots=l_r$, and put $l=l_1$. Let $\theta(\xi)$ be a C^∞-function such that $\theta(\xi)=0$ for $|\xi|\leqq\frac{1}{4}$, and $\theta(\xi)=1$ for $|\xi|\geqq\frac{1}{2}$, and define $\lambda_j(t,x;D_x)$ by

$$\lambda_j(t,x;D_x)\phi=\frac{1}{(2\pi)^n}\int\lambda_j(t,x;\xi)\theta(\xi)\hat{\phi}(\xi)e^{ix\cdot\xi}d\xi.$$

Then $\lambda_j(t,x;D_x)$ are pseudo-differential operators of class $\mathcal{E}_t(S^1)$. Here we have denoted by S^p the set of C^∞-functions $h(x,\xi)$ on $R^n\times R^n$ such that

$$|D_x^\mu D_\xi^\nu h(x,\xi)|\leqq C_{\mu,\nu}(1+|\xi|^2)^{1/2(p-|\nu|)}\qquad \text{in } R^n\times R^n,$$

and $\mathcal{E}_t(S^p)$ the set of infinitely differentiable functions in t valued in S^p. As for the properties of operators of class S^p, see H. Kumano-go [2].

We denote by $\Lambda_k \in \mathcal{E}_t(S^k)$ the space of finite sums of products of at most k elements permitted to choose the same ones from $\{\lambda_j^{(i)}(t, x : D_x)\}_{1 \leq i, j \leq k}$, where $\lambda_j^{(i)}(t, x : \xi) = D_t^i \lambda_j(t, x : \xi)$. Putting $\Pi_m = \partial_1^{l_1} \partial_2^{l_2} \cdots \partial_r^{l_r}$ with $\partial_j = D_t - \lambda_j(t, x : D_x)$ for $1 \leq j \leq r$, we have $P_m - \Pi_m = \sum_{j=0}^{m-1} A_j(t) D_t^{m-j-1}$, where $A_j(t)$ is an element in Λ_j. Define operators δ_j, $0 \leq j \leq m-1$, as follows: $\delta_0 = 1$, $\delta_1 = \partial_1$, $\delta_2 = \partial_2 \partial_1$, $\cdots$, $\delta_{r_1} = \partial_{r_1} \partial_{r_1 - 1} \cdots \partial_2 \partial_1$, $\delta_{r_1+1} = \partial_1 \partial_{r_1}$, $\cdots$, $\delta_{k_1 r_1} = \delta_{r_1} \delta_{(k_1-1) r_1}$, $\cdots$, $\delta_{k_1 r_1 + r_2} = \partial_{r_2} \partial_{r_2 - 1} \cdots \partial_2 \partial_1 \delta_{k_1 r_1}$, $\cdots$, $\delta_{k_1 r_1 + k_2 r_2 + \cdots + k_j r_j} = \delta_{k_j r_j} \cdots \delta_{k_2 r_2} \delta_{k_1 r_1}$, $\cdots$, $\delta_{m-1} = \partial_{r-1} \partial_{r-2} \cdots \partial_2 \partial_1 \delta_{m-r}$, where $k_j = l_{r_j} - l_{r_{j+1}}$. Calculating δ_k, we have easily $D_t^k = \delta_k + \sum_{j=1}^{k} K_j(t) \delta_{k-j}$, $1 \leq k \leq m-1$, where $K_j(t)$ belongs to Λ_j. Thus we obtain

$$P - \Pi_m = \sum_{j=0}^{m-1} B_j(t) \delta_{m-1-j} \quad \text{with} \quad B_j \in \Lambda_j, \tag{2}$$

and the symbols $b_j(t, x ; \xi)$ of $B_j(t)$ are expanded by homogeneous symbols[1)] $c_{j,k}(t, x ; \xi)$ of degree $j-k$

$$b_j(t, x ; \xi) - \sum_{k=0}^{N-1} c_{j,k}(t, x ; \xi) \in \mathcal{E}_t(S^{j-N}) \qquad \text{for any integer } N \geq 1,$$

since $B_j(t)$ belongs to Λ_j. Then we impose the following condition A to P.

Condition A. $c_{m-j,k}(t, x ; \xi) \equiv 0$ *when* $1 \leq j+k \leq m-r$ *and* $0 \leq k \leq l-2$.

We note that this condition A concerns only the homogeneous parts P_j of degree $j, m \leq j \leq m-l+1$, of P and is the same as the one in [4] when $l=2$.

Under this condition we can state the following Theorem 1. The proof of this theorem follows from the energy inequalities and existence of the domain of dependence, which will be shown in sections 3 and 4, respectively.

Theorem 1. *Let* $\{P_m, P_{m-1}, \cdots, P_{m-l+1}\}$ *satisfy the condition A. Then there exists one and only one solution* $u \in C^\infty(\Omega)$ *of the Cauchy problem* (1) *for given* $f \in C^\infty(\Omega)$ *and* $u_j \in C^\infty(R^n)$, $j = 0, 1, \cdots, m-1$.

Throughout this paper we assume the condition A for the differential operator P.

2. Reduction of the operator P. Choose arbitrary k elements $\partial_{j_1}, \partial_{j_2}, \cdots, \partial_{j_k}$, $0 \leq k \leq m-1$, from $\{\underbrace{\partial_1, \partial_1, \cdots, \partial_1}_{l_1}, \underbrace{\partial_2, \partial_2, \cdots, \partial_2}_{l_2}, \cdots, \underbrace{\partial_r, \partial_r, \cdots, \partial_r}_{l_r}\}$ and denote by $\mathfrak{M}$ the set of finite sums of *'monomials'* $\partial_{j_1} \partial_{j_2} \cdots \partial_{j_k}$, $0 \leq k \leq m-1$, with coefficients which are operators of class $\mathcal{E}_t(S^0)$. Then we have the following

1) The homogeneity means the one with respect to ξ for $|\xi| \geq 1$.

Proposition 1. *If $\{P_m, P_{m-1}, \cdots, P_{m-l+1}\}$ satisfies the condition A, then P can be written as $P=\Pi_m+N$ with an operator N in $\mathfrak{M}$.*

Sketch of the proof. In view of the condition A and (2) we have only to prove that $C_{m-j,k}(t)\delta_{j-1}$, $m-r<j+k<m$, belong to $\mathfrak{M}$. Since $\partial_h-\partial_i=\lambda_i(t,x;D_x)-\lambda_h(t,x;D_x)$, $h\neq i$, are elliptic operators, there exist operators $K_{h,i}(t)$ of class $\mathcal{E}_t(S^{-1})$ such that $K_{h,i}(t)(\partial_h-\partial_i)=I-Q_{h,i}(t)$ with operators $Q_{h,i}$ of class $\mathcal{E}_t(S^{-\infty})$. From the construction of δ_{j-1} and the fact $m-j-k<r$, we can find $m-j-k+1$ elements $\partial_{i_1}, \partial_{i_2}, \cdots, \partial_{i_{m-j-k+1}}$ such that these are different each other and $\partial_{\sigma_1}\partial_{\sigma_2}\cdots\partial_{\sigma_{m-j-k}}\delta_{j-1}$ belong to $\mathfrak{M}$ for any $m-j-k$ elements $\partial_{\sigma_1}, \cdots, \partial_{\sigma_{m-j-k}}$ in $\{\partial_{i_1}, \partial_{i_2}, \cdots, \partial_{i_{m-j-k+1}}\}$. Thus it follows that

$$\begin{aligned}
I &= K_{i_1,i_2}(\partial_{i_1}-\partial_{i_2})+Q_{i_1,i_2}\\
&= K_{i_1,i_2}\{K_{i_2,i_3}(\partial_{i_2}-\partial_{i_3})+Q_{i_2,i_3}\}\partial_{i_1}-K_{i_1,i_2}\{K_{i_1,i_3}(\partial_{i_1}-\partial_{i_3})\\
&\quad +Q_{i_1,i_3}\}\partial_{i_2}+Q_{i_1,i_2}\\
&\quad \cdots\\
&= \sum_{(\sigma_1,\cdots,\sigma_{m-j-k})} K_{\sigma_1,\sigma_2,\cdots,\sigma_{m-j-k}}\partial_{\sigma_1}\partial_{\sigma_2}\cdots\partial_{\sigma_{m-j-k}}\\
&\quad +\sum_{p=0}^{m-j-k-1}\sum_{(\sigma_1,\cdots,\sigma_p)} Q_{\sigma_1,\cdots,\sigma_p}\partial_{\sigma_1}\partial_{\sigma_2}\cdots\partial_{\sigma_p},
\end{aligned}$$

where $\sum_{(\sigma_1,\cdots,\sigma_p)}$ means the sum over the permutations of $(\sigma_1, \cdots, \sigma_p)$ and $K_{\sigma_1,\sigma_2,\cdots,\sigma_{m-j-k}}$ are operators of class $\mathcal{E}_t(S^{-(m-j-k)})$ and of class $\mathcal{E}_t(S^{-\infty})$, respectively. Thus $C_{m-j,k}(t)\delta_{j-1}=C_{m-j,k}(t)I\,\delta_{j-1}$ belong to $\mathfrak{M}$. The proof is complete.

3. Theorem 2 (Energy inequalities). *Let p be a real number. Then P satisfies the energy inequality*

$$[E_{(p,l)}]: \sum_{j=0}^{m-l}\|D_t^j u(t,\cdot)\|^2_{(p+m-l-j)}\leqq C(T)\Big\{\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|^2_{(p+m-1-j)}$$
$$+\int_0^t \|(Pu)(t',\cdot)\|^2_{(p)}dt'\Big\}, \quad 0\leqq t\leqq T, \quad u\in C_0^\infty(\Omega),$$

where $\|u(t,\cdot)\|^2_{(p)}=\dfrac{1}{(2\pi)^n}\displaystyle\int(1+|\xi|^2)^p|\hat{u}(t,\xi)|^2d\xi$.

To prove this theorem we begin with

Lemma 1. *Let s denote a 'monomial' in $\mathfrak{M}$, i.e., $s=\partial_{j_1}\partial_{j_2}\cdots\partial_{j_k}$, $0\leqq k\leqq m-1$. Then there exists a constant C independent of u such that*

$$\sum_s\|s[u](t,\cdot)\|^2_{(p)}\leqq C\Big\{\sum_s\|s[u](0,\cdot)\|^2_{(p)}+\int_0^t\|\Pi_m[u](t',\cdot)\|^2_{(p)}dt'\Big\},$$
$$u\in C_0^\infty(\Omega),$$

where $\sum_s$ means the sum of all 'monomials' in $\mathfrak{M}$.

Proof. Let $\langle D_x\rangle^p$ be the pseudo-differential operator with the symbol $(1+|\xi|^2)^{p/2}$, and put $v=\langle D_x\rangle^p(su)$. Considering $\dfrac{d}{dt}\displaystyle\int v(t,x)\overline{v(t,x)}dx$, we have

$$(3)\qquad \|s[u](t,\cdot)\|^2_{(p)}\leqq C\Big\{\|s[u](0,\cdot)\|^2_{(p)}+\int_0^t\|\partial_j s[u](t',\cdot)\|^2_{(p)}dt'\Big\}.$$

Remarking that $\partial_j\partial_k-\partial_k\partial_j=i\{\lambda'_k+\lambda'_j\}+\{\lambda_j\lambda_k-\lambda_k\lambda_j\}=i\{\lambda'_k+\lambda'_j\}\{K_{j,k}(\partial_k-\partial_j)+Q_{k,j}\}+\{\lambda_j\lambda_k-\lambda_k\lambda_j\}\in\mathfrak{M}$, $j\neq k$, we see that for any $s'=\partial_{j_1}\partial_{j_2}\cdots\partial_{j_{m-1}}\in\mathfrak{M}$ there exists a k such that $\Pi_m-\partial_k s'\in\mathfrak{M}$. Thus we have

$$\|s'[u](t,\cdot)\|^2_{(p)}\leqq C\Big\{\|s'[u](0,\cdot)\|^2_{(p)}+\int_0^t\|\Pi_m[u](t',\cdot)\|^2_{(p)}dt'+\sum_s\int_0^t\|s[u](t',\cdot)\|^2_{(p)}dt'\Big\}.$$

From this and (3), it follows that

$$\sum_s\|s[u](t,\cdot)\|^2_{(p)}\leqq C\Big\{\sum_s\|s[u](0,\cdot)\|^2_{(p)}+\int_0^t\|\Pi_m[u](t',\cdot)\|^2_{(p)}dt'+\sum_s\int_0^t\|s[u](t',\cdot)\|^2_{(p)}dt'\Big\}.$$

By Gronwall's inequality, we obtain the desired estimate.

Proof of Theorem 2. Since we can write $P=\Pi_m+N$ with an element N in $\mathfrak{M}$, it follows from Lemma 1 that

$$(4)\qquad \sum_s\|s[u](t,\cdot)\|^2_{(p)}\leqq C\Big\{\sum_s\|s[u]\|^2_{(p)}+\int_0^t\|(Pu)(t',\cdot)\|^2_{(p)}dt'\Big\}.$$

Consider $P_m^{(l-1)}(t,x;\tau,\xi)=(\partial/\partial\tau)^{l-1}\prod_{j=1}^r(\tau-\lambda_j(t,x;\xi))^{l_j}$. Let us denote by Π the pseudo-differential operator obtained by replacing $\tau-\lambda_j(t,x;\xi)$ by ∂_j in $P_m^{(l-1)}$. Then Π belongs to $\mathfrak{M}$ and is a regularly hyperbolic pseudo-differential operator of order $m-l+1$. Consequently, calculating as in the case of a regularly hyperbolic differential operator, we have the energy inequality for Π:

$$(5)\qquad \sum_{j=0}^{m-l}\|D_t^ju(t,\cdot)\|^2_{(p+m-l-j)}\leqq C\Big\{\sum_{j=0}^{m-l}\|D_t^ju(0,\cdot)\|^2_{(p+m-l-j)}+\int_0^t\|\Pi[u](t',\cdot)\|^2_{(p)}dt'\Big\}.$$

On the other hand it follows from $\Pi\in\mathfrak{M}$ that

$$(6)\qquad \|\Pi[u](t,\cdot)\|^2_{(p+m-l-j)}\leqq C\sum_s\|s[u](t,\cdot)\|^2_{(p)}.$$

Integrating both sides of (4) with respect to t, in view of (5) and (6) we obtain the energy inequalities $[E_{(p,l)}]$. Thus the proof is complete.

Let P^* be the adjoint operator of P. Then we can write $P^*=\Pi_m^*+M$, where $\Pi_m^*=\partial_r^{*l_r}\cdots\partial_1^{*l_1}$ with $\partial_j^*=D_t-\lambda_j^*(t,x;D_x)$ and M is a finite sum of $\partial_{j_1}^*\cdots\partial_{j_k}^*$, $0\leqq k\leqq m-1$, with coefficients which are operators of class $\mathcal{E}_t(S^0)$. Thus by the same reasoning as in the case of P, P^* has the energy inequality $[E_{(p,l)}]$.

4. Domain of dependence. Let Γ be the interior of a backward cone: $\{(t,x)\,|\,x-\lambda_0|\leqq\lambda_{\max}|t-t_0|,\ 0<t_0<T\}$, where $\lambda_{\max}=\max_{1\leqq j\leqq r}\sup_{(t,x)\in\Omega,|\xi|=1}|\lambda_j(t,x;\xi)|$. We remark that the condition A is invariant under Holmgren's transformation by making use of Lemma 4.1 of [4]. Thus we deduce the following theorem by a usual argument ([4]).

Theorem 3 (Domain of dependence). *If $u \in C^\infty(\Gamma)$ satisfies that $Pu=0$ in $\Gamma \cap \Omega$ and $D_t^j \mu(0,x)=0$ on $\Gamma \cap \{t=0\}$, $j=0,1,\cdots,m-1$, then u must vanish in $\Gamma \cap \Omega$.*

By Theorems 2 and 3, we obtain Theorem 1.

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