165. On Approximation of Nonlinear Semi-groups

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(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1974)

1. Let X be a real Banach space and let X_0 be a subset of X. By a *contraction semi-group* on X_0, we mean a family $\{T(t); t\geq 0\}$ of operators from X_0 into itself satisfying the following conditions:

(i) $T(0)=I$ (the identity), $T(t+s)=T(t)T(s)$ for $t, s\geq 0$;

(ii) $\|T(t)x-T(t)y\|\leq\|x-y\|$ for $t\geq 0$ and $x, y\in X_0$;

(iii) $\lim_{t\to 0+} T(t)x=x$ for $x\in X_0$.

We define the *infinitesimal generator* A_0 of $\{T(t); t\geq 0\}$ by $A_0x=\lim_{h\to 0+} h^{-1}(T(h)x-x)$, whenever the right side exists.

Throughout this paper, we assume that X_0 is a closed convex set in X and $\{T(t); t\geq 0\}$ is a contraction semi-group on X_0. Let us set

(1.1) $$A_h=h^{-1}(T(h)-I) \qquad \text{for } h>0.$$

Then, for each h, there is the unique contraction semi-group $\{T_h(t); t\geq 0\}$ on X_0, with the infinitesimal generator A_h, and it satisfies

(1.2) $$(d/dt)T_h(t)x=A_hT_h(t)x \qquad \text{for } x\in X_0 \text{ and } t\geq 0.$$

(See Appendix in [10].)

Our purpose is to prove the following theorem.

Theorem. *For each $x\in X_0$, we have*

(1.3) $$T(t)x=\lim_{h\to 0+} T_h(t)x \qquad \text{for } t\geq 0,$$

and the convergence is uniform with respect to t in every bounded interval of $[0,\infty)$.

Remarks. 1) I. Miyadera showed in [9] that the convergence (1.3) holds true for $x\in\bar{E}$, where E is the set of $x\in X_0$ such that $\|A_hx\|$ is bounded as $h\to 0+$. Under the similar conditions, many authors have also treated the convergence (1.3). (See [2], [4], [8] and [10].)

2) This theorem is well known in linear theory. (See [5].)

2. For the proof of Theorem, we shall prepare several lemmas in this section. The following is known.

Lemma 1. *Let $x\in X_0$ and $h>0$. Then for $t>0$,*

(2.1) $$\|A_hT_h(t)x\|\leq\|A_hx\|,$$

(2.2) $$\|T_h(t)x-x\|\leq t\|A_hx\|.$$

Let F be the duality map on X into X^* and we set $\langle x,y\rangle_s=\sup\{\langle x,f\rangle; f\in F(y)\}$ for $x, y\in X$.

Lemma 2. *Let $x, z\in X_0$, $h>0$ and n be a positive integer. Then we have*

(2.3) $$\|z-x\|^2\geq\|T(nh)z-x\|^2+2\sum_{i=1}^n h\langle -A_hx, T(ih)z-x\rangle_s.$$

Proof. By the definition (1.1) of A_h, we have $T(h)x = x + hA_hx$. Therefore

$$\|T((i-1)h)z - x\|^2 \geq \|T(ih)z - T(h)x\|^2 = \|T(ih)z - x - hA_hx\|^2$$
$$\geq \|T(ih)z - x\|^2 + 2h\langle -A_hx, T(ih)z - x\rangle_s,$$

for each $i = 1, 2, \cdots, n$, where we used the inequality: $\|u+v\|^2 \geq \|u\|^2 + 2\langle v, u\rangle_s$. If we add these inequalities for $i = 1, 2, \cdots, n$, we have (2.3). Q.E.D.

Lemma 3. *For each $x \in X_0$, we have*

$$\lim_{(t,h)\to(0,0)} T_h(t)x = x. \tag{2.4}$$

Proof. We mimic M. Crandall and T. Liggett [3] as follows. (See also T. Kato [6].) Let $z, x \in X_0$ and n be a positive integer. Then by Lemma 2, we have

$$\begin{aligned}\|z - T_h(\sigma)x\|^2 \geq{} & \|T(nh)z - T_h(\sigma)x\|^2 \\ & + 2\textstyle\sum_{i=1}^n h\langle -A_hT_h(\sigma)x, T(ih)z - T_h(\sigma)x\rangle_s.\end{aligned} \tag{2.5}$$

We also have

$$\begin{aligned}&\|T(nh)z - T_h(\sigma)x\|^2 \\ &\quad = \|z + nhA_{nh}z - T_h(\sigma)x\|^2 \geq \|z - T_h(\sigma)x\|^2 \\ &\qquad + 2nh\langle A_{nh}z, z - T_h(\sigma)x\rangle_s\end{aligned}$$

and

$$\langle A_{nh}z, z - T_h(\sigma)x\rangle_s \geq -\|A_{nh}z\|(\|z-x\| + \|x - T_h(\sigma)x\|).$$

Thus, combining these inequalities with (2.5), we have

$$\begin{aligned}0 \geq{} & -\|A_{nh}z\|(\|z-x\| + \|x - T_h(\sigma)x\|) \\ & + (2/n)\textstyle\sum_{i=1}^n \langle -A_hT_h(\sigma)x, T(ih)z - T_h(\sigma)x\rangle_s.\end{aligned} \tag{2.6}$$

By T. Kato's lemma, we have

$$(d/d\sigma)\|T(ih)z - T_h(\sigma)x\|^2 = 2\langle -A_hT_h(\sigma)x, T(ih)z - T_h(\sigma)x\rangle_s \tag{2.7}$$

for a.a. $\sigma \geq 0$. Therefore, integrating (2.6) with respect to σ, we have

$$\begin{aligned}0 \geq{} & -2\|A_{nh}z\|\left(t\|z-x\| + \int_0^t \|x - T_h(\sigma)x\|\,d\sigma\right) \\ & + (1/n)\textstyle\sum_{i=1}^n (\|T(ih)z - T_h(t)x\|^2 - \|T(ih)z - x\|^2).\end{aligned} \tag{2.8}$$

But

$$\begin{aligned}&\|T(ih)z - T_h(t)x\|^2 - \|T(ih)z - x\|^2 \\ &\quad \geq -2\|T(ih)z - x\|\,\|x - T_h(t)x\| + \|x - T_h(t)x\|^2\end{aligned}$$

and

$$\begin{aligned}&\|T(ih)z - x\| \leq \|T(ih)z - T(ih)x\| \\ &\quad + \|T(ih)x - x\| \leq \|z - x\| + \|T(ih)x - x\|.\end{aligned}$$

Therefore, (2.8) implies

$$\begin{aligned}\|x - T_h(t)x\|^2 \leq{} & 2(\|z-x\| + (1/n)\textstyle\sum_{i=1}^n \|T(ih)x - x\|)\,\|x - T_h(t)x\| \\ & + 2\|A_{nh}z\|\left(t\|z-x\| + \int_0^t \|x - T_h(\sigma)x\|\,d\sigma\right).\end{aligned}$$

Let us set $z = T_{nh}(t)x$. Then by Lemma 1, it follows that $\|z-x\| \leq t\|A_{nh}x\|$ and $\|A_{nh}z\| \leq \|A_{nh}x\|$. Hence, by simple calculation, we obtain the following estimate:

$$\|x - T_h(t)x\| \leq \eta(t\|A_{nh}x\| + (1/n)\textstyle\sum_{i=1}^n \|T(ih)x - x\|), \tag{2.9}$$

where η is a universal constant. (See Remark after the Proof.)

Now we set $\omega_0(r)=\sup\{\|T(t)x-x\|;\,0\leq t\leq r\}$. Apparently, $\omega_0(r)\to 0$ as $r\to 0+$. Given $\varepsilon>0$, pick $\delta>0$ so that $\omega_0(\delta)<\varepsilon$. Let $0<h\leq\delta/2$. Then $\delta/2\leq nh\leq\delta$ for suitable n. Noting that $A_{nh}x=(T(nh)x-x)/nh$, by (2.9) with this n, we have

$$\|x-T_h(t)x\|\leq\eta(2t/\delta+1)\varepsilon.$$

Hence

$$\|x-T_h(t)x\|\leq 2\eta\varepsilon$$

for $0<h\leq\delta/2$ and $0<t\leq\delta/2$. Q.E.D.

Remark. Put $\varphi(t)=\sup_{0\leq\sigma\leq t}\|x-T_h(\sigma)\|$. Then we have easily

$$\varphi(t)\leq 2(t\,\|A_{nh}x\|+(1/n)\textstyle\sum_{i=1}^n\|T(ih)x-x\|)\varphi(t)$$
$$+2\,\|A_{nh}x\|\,(t^2\,\|A_{nh}x\|+t\varphi(t)).$$

Noting $\|x-T_h(t)x\|\leq\varphi(t)$, we have (2.9) immediately with $\eta\leq 2+\sqrt{6}$.

3. In this section, we give the proof of Theorem. For the purpose, we also consider the family of operators, $\{T([t/h]h);\,t\geq 0\}$, for each $h>0$, where $[t/h]$ is the greatest integer not exceeding t/h.

Let $x\in X_0$ and $T>0$. Lemma 3 shows that $\|T_h(t)x\|$ is uniformly bounded on $[0,T]$ as $h\to 0+$. And apparently, $\|T([t/h]h)x\|$ is uniformly bounded on $[0,T]$. Thus we choose constants M_T and $h_0>0$ so that

$$\|T_h(t)x\|,\,\|T([t/h]h)x\|\leq M_T \tag{3.1}$$

for all $t\in[0,T]$ and $0<h\leq h_0$.

Lemma 4. *Let $0<h\leq h_0$. Then*

$$\begin{aligned}&\int_\alpha^\beta\{\|T([t/h]h)x-T_h(\sigma)x\|^2-\|T([s/h]h)x-T_h(\sigma)x\|^2\}d\sigma\\&\qquad+\int_s^t\{\|T([\tau/h]h)x-T_h(\beta)x\|^2-\|T([\tau/h]h)x-T_h(\alpha)x\|^2\}d\tau\\&\leq 16hM_T^2,\end{aligned}\tag{3.2}$$

for all s,t,α and $\beta\in[0,T]$ such that $s\leq t$ and $\alpha\leq\beta$.

Proof. By Lemma 2, we have

$$\begin{aligned}\|T(mh)x-T_h(\sigma)x\|^2&\geq\|T(nh)x-T_h(\sigma)x\|^2\\&\quad+2\textstyle\sum_{i=m+1}^n\langle -A_hT_h(\sigma)x,\,T(ih)x-T_h(\sigma)x\rangle_s,\end{aligned}\tag{3.3}$$

for $\sigma\in[0,T]$ and integers $n\geq m\geq 0$. Since (2.7) holds true with $z=x$, integrating (3.3) with respect to σ, we have

$$\begin{aligned}0\geq&\int_\alpha^\beta\{\|T(nh)x-T_h(\sigma)x\|^2-\|T(mh)x-T_h(\sigma)x\|^2\}d\sigma\\&+\textstyle\sum_{i=m+1}^n h\{\|T(ih)x-T_h(\beta)x\|^2-\|T(ih)x-T_h(\alpha)x\|^2\}.\end{aligned}$$

Put $n=[t/h]$ and $m=[s/h]$. Then, on account of (3.1), we have (3.2). Q.E.D.

Now, we set $\phi_h(\tau,\sigma):R\times R\to R^+$ by

$$\phi_h(\tau,\sigma)=\begin{cases}\|T([\tau/h]h)x-T_h(\sigma)x\|^2, & \text{for }(\tau,\sigma)\in[0,T]\times[0,T],\\ 0, & \text{otherwise.}\end{cases}$$

Let ρ be a molifier such that $\rho\in\mathcal{D}(R)$, $\rho\geq 0$, supp $[\rho]\subset[-1,1]$ and

$$\int_R \rho(\xi)d\xi=1.$$

And we set $\rho_\varepsilon(\xi,\eta)=\varepsilon^{-2}\rho(\xi/\varepsilon)\rho(\eta/\varepsilon)$ for $\varepsilon>0$. Then we define the regularization of ϕ_h by

$$\phi_{h,\varepsilon}(\tau,\sigma)=(\rho_\varepsilon*\phi_h)(\tau,\sigma)=\iint_{R\times R}\rho_\varepsilon(\xi,\eta)\phi_h(\tau-\xi,\sigma-\eta)d\xi d\eta. \tag{3.4}$$

By Lemma 4, we have immediately

$$\int_\alpha^\beta \{\phi_{h,\varepsilon}(t,\sigma)-\phi_{h,\varepsilon}(s,\sigma)\}d\sigma + \int_s^t \{\phi_{h,\varepsilon}(\tau,\beta)-\phi_{h,\varepsilon}(\tau,\alpha)\}d\tau \le 16hM_T^2$$

or

$$\int_\alpha^\beta \int_s^t \left\{\frac{\partial}{\partial\tau}\phi_{h,\varepsilon}(\tau,\sigma)+\frac{\partial}{\partial\sigma}\phi_{h,\varepsilon}(\tau,\sigma)\right\}d\tau d\sigma \le 16hM_T^2, \tag{3.5}$$

for s, t, α and $\beta \in [\varepsilon, T-\varepsilon]$, $s\le t$, $\alpha\le\beta$.

We set $\omega_0(r)$ as in the proof of Lemma 3 and set

$$\omega_1(r)=\sup\{\|T_h(t)x-x\|;\ 0\le t\le r,\ 0\le h\le r\}.$$

Lemma 3 means that $\omega_1(r)\to 0$ as $r\to 0+$. Also we note that

$$\begin{aligned}&\|T([t/h]h)x-T([s/h]h)x\|\le 3\omega_0(\varepsilon),\\ &\|T_h(t)x-T_h(s)x\|\le\omega_1(\varepsilon),\end{aligned} \tag{3.6}$$

for $0<h\le\varepsilon$ and $t,s\ge 0$ such that $|t-s|\le\varepsilon$.

Lemma 5. *Let ε and δ be sufficiently small. Then we have the following estimate:*

$$\begin{aligned}\phi_h(t,t)-\phi_h(s,s)&\le 8M_T(3\omega_0(\varepsilon)+\omega_1(\varepsilon))\\ &\quad+T(M_{\varepsilon,T}\delta+16hM_T^2\delta^{-2}),\end{aligned} \tag{3.7}$$

for $0<h\le\min(\varepsilon,h_0)$ and $s,t\in[\varepsilon,T-\varepsilon-\delta]$, $s\le t$, where $M_{\varepsilon,T}$ is a non-negative constant, independent of h.

Proof. We follow the argument by T. Takahashi in [11]. (See also Ph. Bénilan [1].) For $r\in[\varepsilon,T-\varepsilon-\delta]$, we set

$$I(r)=\int_r^{r+\delta}\int_r^{r+\delta}\delta^{-2}\left\{\frac{\partial}{\partial\tau}\phi_{h,\varepsilon}(\tau,\sigma)-\frac{\partial}{\partial\tau}\phi_{h,\varepsilon}(r,r)+\frac{\partial}{\partial\sigma}\phi_{h,\varepsilon}(\tau,\sigma)-\frac{\partial}{\partial\sigma}\phi_{h,\varepsilon}(r,r)\right\}d\tau d\sigma.$$

Taking account of (3.1), we see that all partial derivatives of $\phi_{h,\varepsilon}$ up to the second order are uniformly bounded on $[0,T]\times[0,T]$ with respect to $0<h\le h_0$. Hence, for some $M_{\varepsilon,T}>0$ we have

$$|I(r)|\le M_{\varepsilon,T}\delta \qquad \text{for } r\in[\varepsilon,T-\varepsilon-\delta],$$

by mean value theorem. Combing this with (3.5), we have

$$\begin{aligned}\frac{d}{dr}\phi_{h,\varepsilon}(r,r)&=\frac{\partial}{\partial\tau}\phi_{h,\varepsilon}(r,r)+\frac{\partial}{\partial\sigma}\phi_{h,\varepsilon}(r,r)\\ &\le M_{\varepsilon,T}\delta+16hM_T^2\delta^{-2} \qquad \text{for } r\in[\varepsilon,T-\varepsilon-\delta].\end{aligned} \tag{3.8}$$

Thus, integrating (3.8) with respect to r, we obtain

$$\phi_{h,\varepsilon}(t,t)-\phi_{h,\varepsilon}(s,s)\le T(M_{\varepsilon,T}\delta+16hM_T^2\delta^{-2}) \tag{3.9}$$

for $s, t \in [\varepsilon, T-\varepsilon-\delta]$, $s \leq t$.

On the other hand, we have

$$|\phi_{h,\varepsilon}(r, r)-\phi_h(r, r)| \leq \iint_{R\times R} \rho_\varepsilon(\xi, \eta)\, |\phi_h(r-\xi, r-\eta)-\phi_h(r, r)|\, d\xi d\eta$$

and

$$|\phi_h(r-\xi, r-\eta)-\phi_h(r, r)| \leq 4M_T(\|T([(r-\xi)/h]h)x-T([r/h]h)x\|+\|T_h(r-\eta)x-T_h(r)x\|).$$

Therefore, we have by (3.8)

$$(3.10) \qquad |\phi_{h,\varepsilon}(r, r)-\phi_h(r, r)| \leq 4M_T(3\omega_0(\varepsilon)+\omega_1(\varepsilon))$$

for $0<h\leq \min(\varepsilon, h_0)$ and $r \in [\varepsilon, T-\varepsilon]$.

Combining (3.10) with (3.9), we have (3.7). Q.E.D.

Now we can complete the proof of Theorem. Note that

$$\phi_h(\varepsilon, \varepsilon) \leq 2M_T(\|T([\varepsilon/h]h)x-x\|+\|x-T_h(\varepsilon)x\|) \leq 2M_T(2\omega_0(\varepsilon)+\omega_1(\varepsilon))$$

for $0<h\leq \min(\varepsilon, h_0)$. Hence, taking $s=\varepsilon$ in (3.7), we have

$$\phi_h(t, t) \leq 10M_T(3\omega_0(\varepsilon)+\omega_1(\varepsilon))+T(M_{\varepsilon,T}\delta+16hM_T^2\delta^{-2})$$

for $0<h\leq \min(\varepsilon, h_0)$ and $t \in [\varepsilon, T-\varepsilon-\delta]$. Since $M_{\varepsilon,T}$ is independent of h and M_T is independent of both h and ε, this shows that $\|T([t/h]h)x-T_h(t)x\|$ converges to 0 as $h\to 0+$ for every $t \in (0, T)$ and the convergence is uniform on every compact interval of $(0,T)$. But, taking account of (3.6) again, we see that the convergence holds true for every $t \in [0, T]$ and is uniform on $[0, T]$. Since $\|T(t)x-T([t/h]h)x\|\leq \omega_0(h)$, the proof of Theorem is completed.

Remark. After the preparation of this manuscripts, K. Kobayashi [7] proved Theorem by using the following estimate (see Appendix in [10]):

$$\|T_h(t)x-T([t/h]h)x\| \leq (\sqrt{th}+h)\, \|A_h x\| \qquad \text{for } x \in X_0.$$

Acknowledgement. The author would like to express his hearty thanks to Prof. I. Miyadera for valuable advices and personal encouragement.

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