161. On a Theorem of Collingwood

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We denote by D the unit disk $|z|<1$ and by Γ its boundary $|z|=1$. The *cluster set* $C_D(f, e^{i\theta})$ of a complex-valued function $f(z)$ on D at $e^{i\theta}\in\Gamma$ is the set of values α in the extended plane such that $\lim_{n\to\infty} f(z_n)=\alpha$ for a sequence $\{z_n\}$ in D convergent to $e^{i\theta}$. Let $H=\{H(\theta)\}_{e^{i\theta}\in\Gamma}$ be a family of subsets $H(\theta)$ of Γ such that

$$e^{i\theta}\in H'(\theta) \tag{1}$$

for each $e^{i\theta}\in\Gamma$, where $H'(\theta)$ is the derived set of $H(\theta)$. We define the *boundary cluster set* $C_{\Gamma H}(f, e^{i\theta})$ relative to the family H at $e^{i\theta}\in\Gamma$ by

$$C_{\Gamma H}(f, e^{i\theta})=\bigcap_{\eta>0} M_\eta(\theta) \tag{2}$$

where we set

$$M_\eta(\theta)=\overline{\bigcup_{e^{i\theta'}\in H(\theta),0<|\theta'-\theta|<\eta} C_D(f, e^{i\theta'})}. \tag{3}$$

Here and hereafter the bar indicates the closure. We stress that no regularity of $f(z)$ such as holomorphy or continuity is postulated. The purpose of this note is to prove the following

Theorem. *The boundary cluster set $C_{\Gamma H}(f, e^{i\theta})$ coincides with the ordinary cluser set $C_D(f, e^{i\theta})$ for every $e^{i\theta}$ in Γ except possibly for a set of category I in the Baire sense.*

This is originally obtained by Collingwood [1], [2] for a family $H=\{H(\theta)\}_{e^{i\theta}\in\Gamma}$ such that

$$H(\theta)\supset\{e^{i\theta'}; \theta'\in(\theta-\eta,\theta)\} \qquad [\{e^{i\theta'}\in(\theta,\theta+\eta)\}, \text{resp.}] \tag{4}$$

for an $\eta>0$ for every $e^{i\theta}\in\Gamma$. Clearly (4) implies (1), and our theorem generalizes that of Collingwood.

Proof of Theorem. Suppose that $\mathcal{M}(\theta)=\{e^{i\theta}; C_{\Gamma H}(f, e^{i\theta})\neq C_D(f, e^{i\theta})\}$ is of category II. Let $\{\varepsilon_n\}$ be a monotonically decreasing sequence of positive numbers such that $\lim_{n\to\infty}\varepsilon_n=0$. We denote by $C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_n}$ the closed set of points whose spherical distances from $C_{\Gamma H}(f, e^{i\theta})$ are at most equal to ε_n. Let $E_n=\{e^{i\theta}; C_D(f, e^{i\theta})-C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_n}\neq\phi\}$. Obviously $\mathcal{M}(\theta)=\cup E_n$. Hence there exists at least one N such that E_N is of category II. Let $\{\Delta_1, \Delta_2,\cdots,\Delta_m\}$ be a finite triangulation of Riemann-sphere such that each Δ_μ has a spherical diameter less than $\varepsilon_N/4$. For each μ, let $E_{N,\mu}=\{e^{i\theta}\in E_N; \{C_D(f, e^{i\theta})-C_{\Gamma H}(f, e^{i\theta})_{+\varepsilon_N}\}\cap\bar{\Delta}_\mu\neq\phi\}$. Then $E_N=\bigcup_{1\le\mu\le m}E_{N,\mu}$ and therefore there exists at least one M $(1\le M\le m)$ such that $E_{N,M}$ is of category II.

Suppose that $E_{N,M}$ is dense on an arc $(e^{i\theta_1}, e^{i\theta_2})=\{e^{i\theta}; \theta_1<\theta<\theta_2\}$. Choose an $e^{i\theta_0}$ in $E_{N,M}\cap(e^{i\theta_1}, e^{i\theta_2})$. Obviously, $[\varDelta_M, C_{\Gamma H}(f, e^{i\theta_0})]\geq 3\varepsilon_N/4$, where $[\cdot, \cdot]$ denotes the spherical distance. Hence we can easily see that there exists a positive number η such that $[\varDelta_M, M_\eta(\theta_0)]\geq\varepsilon_N/2$, where

$$M\eta(\theta_0)=\overline{\bigcup_{e^{i\theta}\in H(\theta_0), |\theta-\theta_0|<\eta} C_D(f, e^{i\theta})}.$$

Therefore there exist an $e^{i\theta'}\in H(\theta_0)\cap(e^{i(\theta_0-\eta)}, e^{i(\theta_0+\eta)})\cap(e^{i\theta_1}, e^{i\theta_2})$ and a positive number ρ such that $[\varDelta_M, f(z)]\geq\varepsilon_N/3$ for every $z\in D\cap U(e^{i\theta'}, \rho)$, where $U(e^{i\theta'}, \rho)=\{\zeta; |\zeta-e^{i\theta'}|<\rho\}$. Hence for every $e^{i\theta}\in\Gamma\cap U(e^{i\theta'}, \rho)$, $[\varDelta_M, C_D(f, e^{i\theta})]\geq\varepsilon_N/3$. On the other hand, since $E_{N,M}$ is dense on $(e^{i\theta_1}, e^{i\theta_2})\cap U(e^{i\theta'}, \rho)$, there exists an $e^{i\theta''}\in E_{N,M}\cap(e^{i\theta_1}, e^{i\theta_2})\cap U(e^{i\theta'}, \rho)$. For such $e^{i\theta''}$, $\varDelta_M\cap C_D(f, e^{i\theta''})\neq\phi$ by the assumption of $E_{N,M}$. This is a contradiction. Q.E.D.

References

[1] E. F. Collingwood: On sets of maximum indetermination of analytic functions. Math. Zeitcher., **67**, 377–396 (1957).

[2] ——: Addendum: On sets of maximum indetermination of analytic functions. Math. Zeitcher., **68**, 498–499 (1958).