160. The Generalized Form of Poincaré's Inequality and its Application to Hypoellipticity

By Kazuo TANIGUCHI
University of Osaka Prefecture

(Comm. by Kôsaku YOSIDA, M. J. A., Nov. 12, 1974)

Introduction. In this paper we shall derive an inequality of the form

(0.1) $\|u\| \leqq C(\zeta^{-\tau}\|u\|_\tau + \zeta^l \|gu\|)$ for $u \in C_0^\infty(B_{\delta_0})$, $\zeta > 0$

as an extended form of Poincaré's inequality, where B_{δ_0} is the open ball in R_x^n with the center $x=0$ and the radius $\delta_0 > 0$, τ is a positive number, and $g(x)$ is a real valued C^∞-function which vanishes of finite order l at the origin. If g is a homogeneous function satisfying $|g(x)| \geqq C_0 |x|^l$ ($C_0 > 0$) we can easily derive (0.1) by deriving first an inequality $\|u\| \leqq C(\| |D_x|^\tau u\| + \|gu\|)$ and using the homogeneity of g as in Grushin [2]. In the present paper using Hörmander's theorem in [4] we shall prove that the inequality (0.1) holds even in the case of non-homogeneous function $g(x)$.

As an application we shall prove the hypoellipticity for the operator of the form

(0.2) $L = a(X, D_x) + g(X) b(X, Y, D_y)$,

when $a(x, \xi)$ satisfies the conditions similar to those in [3] and [7], $b(x, y, \eta)$ satisfies the conditions similar to those in the strongly elliptic case, and $g(x)$ is a non-negative function such that $\partial_x^{\alpha_0} g(0) \neq 0$ for some α_0. The idea of the proof is found in the proof of the hypoellipticity of the operator $Lu = |x|^2 \Delta_x^2(|x|^2 u) - \Delta_x u + i|x|^2 \Delta_y^3 u$ by Beals [1]. We note that the operator of the form (0.2) is a generalization of the operators $A(x; D_x) + g(x)^2 B(x, y; D_y)$ in Kato [5] and $(-\Delta_x)^l + |x|^{2\nu}(-\Delta_y)^{l'}$ in Grushin [2] and Taniguchi [8].

The author wishes to thank Prof. H. Kumano-go for suggesting these problems and his helpful advice.

§ 1. The generalized form of Poincaré's inequality. In this paper we shall use the following notations:

$$\partial_{x_j} = \partial / \partial x_j, \qquad j = 1, \cdots, n,$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \quad \text{for multi-index } \alpha = (\alpha_1, \cdots, \alpha_n),$$
$$\mathcal{B}(R_x^n) = \{u \in C^\infty(R_x^n);\ \sup_x |\partial_x^\alpha u(x)| < \infty \text{ for any } \alpha\},$$
$$\mathcal{S}(R_x^n) = \{u \in \mathcal{B}(R_x^n);\ x^\alpha \partial_x^\beta u \in \mathcal{B}(R_x^n) \text{ for any } \alpha, \beta\}.$$

Theorem 1. *Let $g(x) \in C^\infty(\overline{B_{\delta_0}})$ be a real valued function which satisfies for some α_0*

(1.1) $$|\partial_x^{\alpha_0} g(x)| \geqq c_0 > 0 \qquad \text{in } B_{\delta_0},$$

(1.1)′ $$\partial_x^{\beta} g(0) = 0 \qquad \text{for } |\beta| < |\alpha_0|,$$

where B_{δ_0} is an open ball in R_x^n with the center $x=0$ and the radius $\delta_0 (>0)$. Then we have for $\tau>0$

(1.2) $$\|u\| \leqq C(\zeta^{-\tau} \|u\|_\tau + \zeta^{|\alpha_0|} \|gu\|) \qquad \text{for } u \in C_0^\infty(B_{\delta_0}),\ \zeta>0.$$

Remark. In (1.2) setting $\zeta = c\delta^{-1}$ for small constant c we can easily prove Poincaré's inequality

$$\|u\| \leqq C\delta^\tau \|u\|_\tau \qquad \text{for } u \in C_0^\infty(B_\delta),\ 0<\delta<\delta_0,$$

since we have $|g(x)| \leqq C_1 |x|^{|\alpha_0|}$ for a constant C_1.

Proof. As in [4] we use the notations e^{tX}, $|v|_{X,s}$ for a vector field X in $\Omega = B_{\delta_0} \times R_y^1$ and $0<s\leqq 1$ as follows:

e^{tX}: one parameter group of transformations in Ω defined by X,

$$|v|_{X,s} = \sup_{0<t\leqq 1} t^{-s} \|e^{tX} v - v\|_{L^2_{x,y}}, \qquad \text{where } L^2_{x,y} = L^2(R_x^n \times R_y^1).$$

First we assume $0<\tau\leqq 1$ and prove the next inequality

(1.2)′ $$\zeta^{\tau_1} \|u\| \leqq C(\|u\|_\tau + \zeta \|gu\|) \qquad (\tau_1 = (1+|\alpha_0|/\tau)^{-1})$$

which is equivalent to (1.2). Moreover we may assume $\zeta \geqq C_0$ for some constant $C_0>0$ in (1.2)′, since (1.2)′ is trivial for $0<\zeta\leqq C_0$. We put $X_0 = g(x)\partial_y$, $X_1 = \partial_{x_1}, \cdots, X_n = \partial_{x_n}$, $s_0 = 1$, $s_1 = \cdots = s_n = \tau$. Then we have for $Y = \partial_y$

$$Y = (\partial_x^{\alpha_0} g(x))^{-1} (\text{ad } X_1)^{\alpha_{01}} (\text{ad } X_2)^{\alpha_{02}} \cdots (\text{ad } X_n)^{\alpha_{0n}} X_0$$
$$((\text{ad } X)Y = XY - YX,\ \alpha_0 = (\alpha_{01}, \cdots, \alpha_{0n}))$$

and we have the next formula by Theorem 4.3 in [4]

(1.3) $$|v|_{Y,\tau_1} \leqq C_1 \Big(\sum_{j=1}^n |v|_{X_j,\tau} + |v|_{X_0,1} + \|v\| \Big) \qquad \text{for } v \in C_0^\infty(B_{\delta_0} \times \{y\,;\, |y|<1\}).$$

We fix a function $\chi(y) \in C_0^\infty((-1,1))$ such that $\chi \geqq 0$ and $\int \chi(y)^2 dy = 1$, and put $v_\zeta(x,y) = \chi(y) e^{i\zeta y} u(x)$ for $u \in C_0^\infty(B_{\delta_0})$. Then we have from (1.3)

(1.4) $$|v_\zeta|_{Y,\tau_1} \leqq C_1 \Big(\sum_{j=1}^n |v_\zeta|_{X_j,\tau} + |v_\zeta|_{X_0,1} + \|v_\zeta\| \Big).$$

We calculate each term. To begin with we have

(1.5) $$\|v_\zeta\|^2 = \iint |\chi(y) e^{i\zeta y} u(x)|^2 \, dxdy = \|u\|^2 \leqq C_2 \|u\|_\tau^2.$$

Since $(e^{tX_j} v)(x,y) = v(x+te_j, y)$ $(e_j = (0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0))$ for $j \geqq 1$, we have

(1.6) $$|v_\zeta|_{X_j,\tau} = \sup_{0<t\leqq 1} \Big\{ t^{-2\tau} \iint |\chi(y)|^2 |u(x+te_j) - u(x)|^2 \, dxdy \Big\}^{1/2}$$
$$= \sup_{0<t\leqq 1} \Big\{ t^{-2\tau} \int |u(x+te_j) - u(x)|^2 \, dx \Big\}^{1/2} \leqq C_3 \|u\|_\tau.$$

Next we have from $(e^{tX_0} v_\zeta)(x,y) = v_\zeta(x, y+tg(x))$

$$t^{-2} \|e^{tX_0} v_\zeta - v_\zeta\|^2$$
$$= t^{-2} \iint |\chi(y+tg(x)) e^{i\zeta(y+tg(x))} u(x) - \chi(y) e^{i\zeta y} u(x)|^2 \, dxdy$$

$$=\int |g(x)u(x)|^2\,dx\int\left|\int_0^1\{\chi'(y+\theta tg(x))e^{i\zeta(y+\theta tg(x))}+\chi(y+\theta tg(x))i\zeta e^{i\zeta(y+\theta tg(x))}\}d\theta\right|^2 dy$$

$$\leqq C_4^2\zeta^2\,\|gu\|^2 \qquad (\zeta\geqq C_0).$$

Then we get

$$|v_\zeta|_{X_0,1}\leqq C_4\zeta\,\|gu\|. \tag{1.7}$$

Similarly we have

$$\begin{aligned}|v_\zeta|^2_{Y,\tau_1}&=\sup_{0<t\leqq 1} t^{-2\tau_1}\,\|e^{tY}v_\zeta-v_\zeta\|^2\\ &=\sup_{0<t\leqq 1} t^{-2\tau_1}\iint |\chi(y+t)e^{i\zeta(y+t)}u(x)-\chi(y)e^{i\zeta y}u(x)|^2\,dxdy\\ &\geqq \|u\|^2\sup_{0<t\leqq 1} t^{-2\tau_1}\int\left\{\frac{1}{2}|\chi(y)|^2\,|e^{i\zeta(y+t)}-e^{i\zeta y}|^2-|\chi(y+t)-\chi(y)|^2\,|e^{i\zeta(y+t)}|^2\right\}dy\\ &\geqq C_5\zeta^{2\tau_1}\,\|u\|^2-C_6\,\|u\|^2 \qquad (\zeta\geqq C_0).\end{aligned} \tag{1.8}$$

Therefore we have (1.2)′ from (1.4)–(1.8). For $\tau\geqq 1$ we can prove (1.2) by interpolation and (1.2) for $0<\tau\leqq 1$.

§ 2. Hypoellipticity at the origin. In this section we shall study a scalar differential operator in $R^n_x\times R^k_y$ of the form

$$L(X,Y,D_x,D_y)=a(X,D_x)+g(X)b(X,Y,D_y). \tag{2.1}$$

We say that L is hypoelliptic at the origin if there exists a neighborhood Ω of the origin such that $Lu\in C^\infty(\Omega')$ implies $u\in C^\infty(\Omega')$ for $u\in\mathcal{D}'(\Omega)$ and any open set Ω' in Ω.

Before the formulation we introduce some notations.

Notations. Let $\lambda(\xi)$, $\mu(\eta)$ be C^∞-functions in R^n_ξ, R^k_η, respectively, such that for $0<\sigma\leqq 1$

$$(1+|\xi|)^\sigma\leqq\lambda(\xi)\leqq C(1+|\xi|),\qquad (1+|\eta|)^\sigma\leqq\mu(\eta)\leqq C'(1+|\eta|), \tag{2.2}$$

$$|\partial_\xi^\alpha\lambda(\xi)|\leqq C_\alpha\lambda(\xi)^{1-|\alpha|},\qquad |\partial_\eta^{\alpha'}\mu(\eta)|\leqq C_{\alpha'}\mu(\eta)^{1-|\alpha'|}. \tag{2.3}$$

1°) $S^m_{\lambda,1,\delta}=\{p(x,\xi)\in C^\infty(R^{2n}_{x,\xi})\,;\,|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\leqq C_{\alpha\beta}\lambda(\xi)^{m-|\alpha|+\delta|\beta|}\}$ $(-\infty<m<\infty,\ 0\leqq\delta<1)$.

$S^{-\infty}_\lambda=\bigcap_m S^m_{\lambda,1,\delta}$ (cf. [3], [6] and [8]).

2°) $\mathcal{B}_x(S^{m'}_\mu)=\{q(x,y,\eta)\in C^\infty(R^n_x\times R^{2k}_{y,\eta})\,;\,|\partial_x^\gamma\partial_\eta^{\alpha'}\partial_y^{\beta'}q(x,y,\eta)|\leqq C_{\alpha'\beta'\gamma}\mu(\eta)^{m'-|\alpha'|}\}$ $(-\infty<m'<\infty)$.

3°) For $p(x,\xi)\in S^m_{\lambda,1,\delta}$ and $q(x,y,\eta)\in\mathcal{B}_x(S^{m'}_\mu)$ we define pseudo-differential operators $P=p(X,D_x)$, $Q=q(X,Y,D_y)$ with symbols $\sigma(P)(x,\xi)=p(x,\xi)$, $\sigma(Q)(x,y,\eta)=q(x,y,\eta)$ by

$$Pv=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\left(\int e^{-ix\cdot\xi}v(x)dx\right)d\xi,$$

$$Pu=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\left(\int e^{-ix\cdot\xi}u(x,y)dx\right)d\xi,$$

$$Qu=(2\pi)^{-k}\int e^{iy\cdot\eta}q(x,y,\eta)\left(\int e^{-iy\cdot\eta}u(x,y)dy\right)d\eta,$$

for $v\in\mathcal{S}(R^n_x)$ and $u\in\mathcal{S}(R^{n+k}_{x,y})$.

4°) For $P=p(X,D_x)\in S^m_{\lambda,1,\delta}$ we denote the formal adjoint of P by $P^{(*)}=p^{(*)}(X,D_x)$, which is defined by

$$(Pu,v)=(u,P^{(*)}v) \qquad \text{for } u,v\in\mathcal{S}(R^n_x).$$

Conditions. 1) $a(x,\xi)$ belongs to $S^m_{\lambda,1,0}$ $(m>0)$ and satisfies for large $|\xi|$

$$\operatorname{Re} a(x,\xi)\geqq C_0\lambda(\xi)^{\tau m} \qquad (0<\tau\leqq 1,\ C_0>0), \tag{2.4}$$

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)/\operatorname{Re} a(x,\xi)|\leqq C_{\alpha\beta}\lambda(\xi)^{-|\alpha|+\delta|\beta|} \qquad (0\leqq\delta<1) \tag{2.5}$$

(cf. [3], p. 164 and [7], p. 154).

2) $b(x,y,\eta)$ belongs to $\mathcal{B}_x(S^{m'}_\mu)$ $(m'>0)$ and there exists $b_0(x,y,\eta)\in\mathcal{B}_x(S^{m'}_\mu)$ such that

$$b(x,y,\eta)-b_0(x,y,\eta)\in\mathcal{B}_x(S^{m'-1}_\mu)$$

and for large $|\eta|$

$$|b_0(x,y,\eta)|\geqq C'_0\mu(\eta)^{m'} \qquad (C'_0>0) \tag{2.6}$$

$$\operatorname{Re} b_0(x,y,\eta)\geqq 0. \tag{2.7}$$

3) $g(x)$ belongs to $\mathcal{B}(R^n_x)$, $g(x)\geqq 0$ and for some α_0

$$\partial_x^{\alpha_0}g(0)\neq 0. \tag{2.8}$$

Theorem 2. *Under the conditions above the operator* (2.1) *is hypoelliptic at the origin.*

Lemma. *We put* $p(x,\xi)=(1/2)(a(x,\xi)+a^{(*)}(x,\xi))$. *Then* $p(x,\xi)$ *has a fractional power* $\{p_t\}_{t\in R}$ *such that*

$$\begin{cases} p_t\in S^{mt}_{\lambda,1,\delta}, & |p_t(x,\xi)|\geqq C\lambda(\xi)^{\tau mt} \text{ for large } |\xi| \quad (t\geqq 0)\\ p_t\in S^{\tau mt}_{\lambda,1,\delta}, & |p_t(x,\xi)|\geqq C'\lambda(\xi)^{mt} \text{ for large } |\xi| \quad (t<0). \end{cases} \tag{2.9}$$

$$P_0=I \text{ (identity operator)}, \qquad P_1=P \text{ (original operator)}. \tag{2.10}$$

$$|\partial_\xi^\alpha\partial_x^\beta p_t(x,\xi)/p_t(x,\xi)|\leqq C_{\alpha\beta}\lambda(\xi)^{-|\alpha|+\delta|\beta|} \qquad \text{for large } |\xi|. \tag{2.11}$$

$$\sigma(P_{t_1}P_{t_2})-p_{t_1+t_2}\in S^{-\infty}_\lambda, \qquad p_t^{(*)}-p_t\in S^{-\infty}_\lambda. \tag{2.12}$$

Proof is carried out by the similar way to that in [7].

Here we introduce three Sobolev spaces.

$$H_{t,s}=\{u\in\mathcal{S}'(R^{n+k}_{x,y})\,;\,\lambda(D_x)^t\mu(D_y)^s u\in L^2\}$$

with the norm $\|u\|_{t,s}=\|\lambda(D_x)^t\mu(D_y)^s u\|_{L^2_{x,y}}$.

$$\mathcal{H}_{t,s}=\{u\in\textstyle\bigcup_{t'}H_{t',s}\,;\,P_t\mu(D_y)^s u\in L^2\}$$

with the norm

$$\|u\|_{t,s,P}=\{\|P_t u\|^2_{0,s}+\|\Phi(D_x)u\|^2_{0,s}\}^{1/2} \tag{2.13}$$

where $\Phi(\xi)$ is a fixed function of $\mathcal{S}(R^n_\xi)$ such that $\Phi(\xi)>0$ in R^n_ξ (cf. §4 of [7]).

$$W_s=\{u\in\mathcal{H}_{\frac{1}{2},s}\,;\,gu\in\mathcal{H}_{-\frac{1}{2},s+m'}\}$$

with the norm $|||u|||_s=\{\|u\|^2_{\frac{1}{2},s,P}+\|gu\|^2_{-\frac{1}{2},s+m',P}\}^{1/2}$ (cf. [1]).

Let ω be a neighborhood of the origin in R^n_x such that

$$|\partial_x^{\alpha_0}g(x)|\geqq c_0>0 \qquad \text{on } \bar{\omega}, \tag{2.14}$$

which is guaranteed by (2.8). Then we have

Proposition 1. *For* $s\in R^1$ *and* $0\leqq t\leqq 1$ *there exists a constant* C *such that*

$$\|u\|_{t/2,s+\rho_0(1-t),P}\leqq C\,|||u|||_s \qquad \text{for } u\in C_0^\infty(\Omega) \tag{2.15}$$

where $\rho_0=\sigma\tau mm'/2(\sigma\tau m+2|\alpha_0|)$ *and* $\Omega=\omega\times R^k_y$.

Proof. From Theorem 1 and (2.14) we have for $\tau_1=(\sigma\tau m+2|\alpha_0|)/\sigma\tau m$

(2.16) $$\zeta\|v\|^2\leqq C_1(\|v\|^2_{\frac{1}{2}\sigma\tau m}+\zeta^{\tau_1}\|gv\|^2) \qquad \text{for } v\in C_0^\infty(\omega), \zeta>0.$$

Since we can write $I=P_{\frac{1}{2}}^{(*)}P_{-\frac{1}{2}}+R$ $(R\in S_\lambda^{-\infty})$ from (2.12), we have

(2.17) $$\begin{aligned}\|gv\|^2&\leqq C_2\|\sqrt{g}\,v\|^2=C_2(gv,v)\\ &=C_2\{(P_{-\frac{1}{2}}gv,P_{\frac{1}{2}}v)+(Rgv,v)\}\\ &\leqq C_3\{\zeta^{\tau_1}\|P_{-\frac{1}{2}}gv\|^2+\zeta^{-\tau_1}\|P_{\frac{1}{2}}v\|^2+\zeta^{\tau_1}\|\lambda(D_x)^{-\frac{1}{2}m}gv\|^2\\ &\qquad+\zeta^{-\tau_1}\|\lambda(D_x)^{\frac{1}{2}\tau m}v\|^2\}.\end{aligned}$$

Noting (2.2) we have from (2.16) and (2.17)

(2.18) $$\zeta\|v\|^2\leqq C_4\{\|P_{\frac{1}{2}}v\|^2+\|\lambda(D_x)^{\frac{1}{2}\tau m}v\|^2+\zeta^{2\tau_1}(\|P_{-\frac{1}{2}}gv\|^2+\|\lambda(D_x)^{-\frac{1}{2}m}gv\|^2)\}.$$

We denote for Φ used in (2.13)

$$\|v\|_{t,P}=\{\|P_tv\|^2+\|\Phi(D_x)v\|^2\}^{1/2}.$$

Then we have as Theorem 4.1 in [7]

$$\|\lambda(D_x)^{\frac{1}{2}\tau m}v\|\leqq C_5\|v\|_{\frac{1}{2},P}, \qquad \|\lambda(D_x)^{-\frac{1}{2}m}v\|\leqq C_6\|v\|_{-\frac{1}{2},P}$$

and we get from (2.18)

(2.19) $$\zeta\|v\|^2\leqq C_7(\|v\|^2_{\frac{1}{2},P}+\zeta^{2\tau_1}\|gv\|^2_{-\frac{1}{2},P}).$$

Using this and Friedrichs parts as in [6] with respect to

$$\zeta^{(1-t)}(\operatorname{Re}a(x,\xi)+\psi(\xi))^t\leqq C((\operatorname{Re}a(x,\xi)+\psi(\xi))+\zeta) \qquad (0\leqq t\leqq 1)$$

for some $\psi(\xi)\in C_0^\infty(R^n_\xi)$ such that $\operatorname{Re}a(x,\xi)+\psi(\xi)\geqq 0$ for all ξ, we can get for $0\leqq t\leqq 1$

(2.20) $$\zeta^{(1-t)}\|v\|^2_{t/2,P}\leqq C_8(\|v\|^2_{\frac{1}{2},P}+\zeta^{2\tau_1}\|gv\|^2_{-\frac{1}{2},P}) \qquad \text{for } v\in C_0^\infty(\omega),\ \xi>0.$$

Writing $\tilde{u}(x,\eta)=\int e^{-iy\cdot\eta}u(x,y)dy$, we have

$$\|u\|^2_{t,s,P}=(2\pi)^{-k}\int \mu(\eta)^{2s}\|\tilde{u}(\cdot,\eta)\|^2_{t,P}\,d\eta.$$

By putting $\zeta=\mu(\eta)^{2\rho_0}$ in (2.20) we have (2.15) as follows:

$$\begin{aligned}\|u\|^2_{t/2,s+\rho_0(1-t),P}&=(2\pi)^{-k}\int \mu(\eta)^{2s}\zeta^{(1-t)}\|\tilde{u}\|^2_{t/2,P}\,d\eta\\ &\leqq(2\pi)^{-k}C_8\int \mu(\eta)^{2s}\{\|\tilde{u}\|^2_{\frac{1}{2},P}+\zeta^{2\tau_1}\|g\tilde{u}\|^2_{-\frac{1}{2},P}\}d\eta\\ &=C_8|||u|||^2_s.\end{aligned}$$

Here we use the fact that $\zeta^{2\tau_1}=\mu(\eta)^{2m'}$.

Proposition 2. *For any integer* $l(\geqq 0)$, *and real numbers* s, s_1, t_1, *there exists a constant* C *such that*

(2.21) $$\begin{aligned}&\|u\|_{l+\frac{1}{2},s-lm',P}+\|gu\|_{l-\frac{1}{2},s+m'-lm',P}\\ &\qquad\leqq C(\|Lu\|_{l-\frac{1}{2},s,P}+\|u\|_{t_1,s_1}) \qquad \text{for } u\in C_0^\infty(\Omega).\end{aligned}$$

Proof is omitted.

Using Propositions 1 and 2 we can prove that for any open set Ω' in Ω, integer $l(\geqq 0)$, real number s, and $u\in\mathcal{D}'(\Omega)$, $Lu\in\mathcal{H}^{\mathrm{loc}}_{l-\frac{1}{2},s}(\Omega')$ implies $u\in\mathcal{H}^{\mathrm{loc}}_{l+\frac{1}{2},s-lm'}(\Omega')$. Then Theorem 2 is proved. The detailed proof will be published elsewhere.

References

[1] R. Beals: Spatially inhomogeneous pseudodifferential operators. III (to appear).

[2] V. V. Grushin: On a class of hypoelliptic operators. Math. USSR Sb., **12**, 458–476 (1970).

[3] L. Hörmander: Pseudo-differential operators and hypoelliptic equations. Proc. Symposium on Singular Integrals. Amer. Math. Soc., **10**, 138–183 (1967).

[4] ——: Hypoelliptic second order differential equations. Acta. Math., **119**, 147–171 (1967).

[5] Y. Kato: On a class of hypoelliptic differential operators. Proc. Japan Acad., **46**, 33–37 (1970).

[6] H. Kumano-go: Algebras of pseudo-differential operators. J. Fac. Sci. Univ. Tokyo, **17**, 31–50 (1970).

[7] H. Kumano-go and C. Tsutsumi: Complex powers of hypoelliptic pseudo-differential operators with applications. Osaka J. Math., **10**, 147–174 (1973).

[8] K. Taniguchi: On the hypoellipticity and the global analytic-hypoellipticity of pseudo-differential operators. Osaka J. Math., **11**, 221–238 (1974).