159. On the Prolongation of Local Holomorphic Solutions of Nonlinear Partial Differential Equations

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1. Introduction. Holomorphic continuation of solutions of linear partial differential equations in the complex domain has been the subject of recent investigations. (See e.g. Zerner [4] and Tsuno [3].) One of the fundamental results regarding this subject is a theorem of Zerner [4] which asserts that the solutions of a linear partial differential equation can be continued holomorphically over any non-characteristic hypersurfaces. The main purpose of this note is to present an analogous continuation theorem for general nonlinear partial differential equations. The question of the existence of noncontinuable holomorphic solutions is also studied. The complete proofs of our results will be published elsewhere.

2. The Cauchy-Kowalewsky theorem. We consider the following nonlinear system of equations for unknown functions $u_1(z), \cdots, u_N(z)$ in the complex n-space C^n:

$$\frac{\partial^m u_j}{\partial z_1^m} = f_j\left(z_1, \cdots, z_n, \cdots, \left(\frac{\partial}{\partial z}\right)^\alpha u_k, \cdots\right) \qquad j=1, \cdots, N, \tag{1}$$

where f_j depends on the variables $z=(z_1, \cdots, z_n)$ and $(\partial/\partial z)^\alpha u_k(z)$ with multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $|\alpha| \leqq m$, $\alpha_1 \leqq m-1$ and $k=1, \cdots, N$. We impose the initial conditions on $u_j(z)$ on the complex hyperplane $z_1=0$ as follows.

$$\begin{cases} u_j(0, z') = \phi_{j,0}(z') \\ \cdots \\ \dfrac{\partial^{m-1} u_j}{\partial z_1^{m-1}}(0, z') = \phi_{j,m-1}(z') \end{cases} \qquad j=1, \cdots, N, \tag{2}$$

where $z'=(z_2, \cdots, z_n)$ and $\phi_{j,k}(z')$ are arbitrarily given functions. For the regularity of f_j and $\phi_{j,k}$, we suppose that

(i) $f_j(z_1, \cdots, z_n, \cdots, p_{k,\alpha}, \cdots)$, where the variables $p_{k,\alpha}$ stand for the terms $(\partial/\partial z)^\alpha u_k$, are holomorphic on a closed polydisc $|z_\nu| \leqq r$ $(\nu=1, \cdots, n)$, $|p_{k,\alpha}| < \infty$,

(ii) $\phi_{j,k}(z')$ are holomorphic on $|z_\nu| \leqq r$ $(\nu=2, \cdots, n)$,

and set

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*) Partially supported by Fûjukai.

$$C=\max_{j,k}\left\{\left|\left(\frac{\partial}{\partial z'}\right)^{\alpha'}\phi_{j,k}(z')\right|\,\Big|\,|z_\nu|\leqq r,\ |\alpha'|+k\leqq m+1\right\}$$

$$M=\max\left\{1,\ |f_j(z,p)|,\ \left|\frac{\partial f_j}{\partial z_k}\right|,\ \left|\frac{\partial f_j}{\partial p_{k,\alpha}}\right|\,\Big|\,|z_\nu|\leqq r,\ |p_{k,\alpha}|\leqq C+r\right\}$$

and $\hat{N}=(m+n)!/(m!\,n!)$ the cardinal number of the set of all multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leqq m$.

Under these conditions, the classical Cauchy-Kowalewsky theorem ensures the existence of a unique solution of the initial value problem (1)–(2) which is holomorphic in a neighborhood of the origin. (See e.g. Courant-Hilbert [1] and F. John [2].) A quantitative information on the domain in which the solution of (1)–(2) actually exists is given in the following

Theorem 1. *There exists a unique solution* $(u_1(z),\cdots,u_N(z))$ *of the initial value problem* (1)–(2) *in the following domain*:

$$|z_1|<r/\{4\hat{M}(N\hat{N}+1)n\},\qquad |z_2|+\cdots+|z_n|<r/4,$$

where $\hat{M}=3(1+r+C)(N\hat{N}M)^2$.

3. Holomorphic continuation. In this section we state continuation theorems which can be obtained with the aid of Theorem 1. In this and the following sections, we denote by U some neighborhood of 0 in C^n and by Ω some domain in U with regular boundary $\partial\Omega$ containing the origin. We take the coordinates $(z_1,\cdots,z_n)$ so that the complex tangent plane of $\partial\Omega$ at 0 is $z_1=0$. This means that there exists a real-valued C^1 function $\phi(z)$ in U such that $\phi(0)=0$, $d\phi\neq 0$, $\Omega=\{z\in U\,|\,\phi(z)<0\}$ and $\left(\frac{\partial\phi}{\partial z_1}(0),\cdots,\frac{\partial\phi}{\partial z_n}(0)\right)=(1,0,\cdots,0)$.

Definition. A holomorphic function $u(z)$ in Ω is said to be *bounded of order m* if $u(z)$ and all its derivatives of order less than or equal to m are bounded in Ω.

We first consider the quasi-linear system of equations for N unknown functions $u_1(z),\cdots,u_N(z)$.

$$\sum_{|\alpha|=m}\sum_{k=1}^{N}a_\alpha^{j,k}\left(z,\cdots,\left(\frac{\partial}{\partial z}\right)^\beta u_l,\cdots\right)\left(\frac{\partial}{\partial z}\right)^\alpha u_k(z)$$
$$(3)\qquad =f_j\left(z,\cdots,\left(\frac{\partial}{\partial z}\right)^\beta u_l\cdots\right),\qquad j=1,\cdots,N,$$

where $a_\alpha^{j,k}$ and f_j depend on the variables $z_1,\cdots,z_n$ and $p_{l,\beta}=\left(\frac{\partial}{\partial z}\right)^\beta u_l$ $(l=1,\cdots,N,\ |\beta|\leqq m-1)$. We suppose that they are holomorphic in $z\in U$ and $|p_{l,\beta}|<\infty$, and make the following condition A, which implies that the plane $\{z_1=\text{const.}\}$ is non-characteristic for any Cauchy data.

Condition A: *for all* $(z,\cdots,p_{l,\beta},\cdots)$,

$$\det\left(a^{j,k}_{(m,0,\cdots,0)}(z,p)\right)\neq 0.$$

Then we have the following theorem.

Theorem 2. *Under the condition* A, *every solution of* (3) *in* Ω *which is bounded of order* $m+1$ *becomes holomorphic near the origin.*

We remark that when the system (3) consists of a single linear equation, the condition A means that the plane $\{z_1=\text{const.}\}$ is non-characteristic. Therefore this theorem is a partial extension of Zerner's theorem [4].

We next study the system of general nonlinear equations of order m:

$$(4) \qquad F_j\left(z_1, \cdots, z_n, \cdots, \left(\frac{\partial}{\partial z}\right)^\alpha u_k, \cdots\right)=0 \qquad j=1, \cdots, N$$

where F_j depend on $z_1, \cdots, z_n$ and $p_{k,\alpha}=\left(\frac{\partial}{\partial z}\right)^\alpha u_k$ with $|\alpha|\leqq m$, $k=1, \cdots, N$, and are holomorphic in $z \in U$ and $|p_{k,\alpha}|<\infty$. If we differentiate the equation (4) with respect to z_1, then (4) is reduced to a quasi-linear system of order $m+1$. Hence we have the next corollary.

Corollary. *Under the condition* B *below, every solution of* (4) *in* Ω *which is bounded of order* $m+2$ *becomes holomorphic near the origin.*

Condition B: *for all* $(z, \cdots, p_{l,\beta}, \cdots)$,

$$\det\left(\frac{\partial F_j}{\partial p_{k,(m,0,\cdots,0)}}(z, p)\right)\neq 0.$$

That the boundedness assumptions as required in Theorem 2 and Corollary are necessary is shown in the following example.

Example. The equation in C^n

$$\frac{\partial u}{\partial z_1}+u^2=0$$

has a solution $u(z)=z_1^{-1}$.

4. Existence of non-continuable solutions. Lastly we study a single equation with two independent variables for which the condition B is not satisfied. We denote the independent variables by (x, y) instead of (z_1, z_2) and write $p=u_x=\partial u/\partial x$, $q=u_y=\partial u/\partial y$. The equation to be considered is

$$(5) \qquad F(x, y, u, u_x, u_y)=0,$$

where $F(x, y, u, p, q)$ is holomorphic in $U\times C^3$. We now suppose that $F(0, 0, u_0, p_0, q_0)=0$ and $F_p(0, 0, u_0, p_0, q_0)=0$ for some u_0, p_0, q_0.

We may assume, without loss of generality, that $u_0=p_0=q_0=0$, since otherwise we could introduce $u(x, y)-u_0-p_0x-q_0y$ as a new dependent variable. Moreover we assume that $F_q(0, \cdots, 0)=1$. Then the hyperplane $\{y=0\}$ is non-characteristic for small Cauchy data and the characteristic curve $(x_0(t), y_0(t))$ through $(0, 0)$ at $t=0$ is non-singular for small t. By solving the Cauchy problem for (5) with suitable Cauchy data on $\{y=0\}$ which have singularities, we have the next theorem.

Theorem 3. *For the equation* (5) *we assume that* $F(0, \cdots, 0)=0$, $F_p(0, \cdots, 0)=0$ *and* $F_q(0, \cdots, 0)=1$. *Then under the condition* C *below, there exists a solution of* (5) *which is holomorphic in* $\{(x, y) \in V \mid \phi(x, y) < 0\}$ *but cannot be holomorphic at* 0, *where* V *is a small neighborhood of* 0 *and* $\phi(x, y)$ *is of class* C^2.

Condition C: *Let* $(x_0(t), y_0(t), u_0(t), p_0(t), q_0(t))$ *be a characteristic strip of* (5) *through* $(0, \cdots, 0)$ *at* $t=0$, *then for every* $t_0 \neq 0$ $(t_0 \in C)$ *and a real parameter* τ,

$$\frac{d^2}{d\tau^2}\phi(x_0(\tau t_0), y_0(\tau t_0))\big|_{\tau=0} > 0.$$

We remark that by this condition C, Ω becomes strictly pseudo-convex at 0.

References

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