187. *Denseness of Singular Densities*

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Consider a 2-form $P(z)dxdy$ on an open Riemann surface R such that the coefficients $P(z)$ are nonnegative locally Hölder continuous functions of local parameters $z=x+iy$ on R. Such a 2-form $P(z)dxdy$ will be referred to as a *density* on R. We shall call a density P *singular* if any nonnegative C^2 solution u of the elliptic equation

(1) $\Delta u(z)=P(z)u(z)$ (i.e. $d^*du(z)=u(z)P(z)dxdy$)

on R has the zero infimum, i.e. $\inf_{z\in R} u(z)=0$. We denote by $D=D(R)$ and $D_S=D_S(R)$ the set of densities and singular densities on R, respectively. According to Myrberg [2], (1) always possesses at least one strictly positive solution for any open Riemann surface R. In connection with the existence of Evans solution, Nakai [5] showed that $D_S\neq\emptyset$ for any open Riemann surface R. The purpose of this note is to show that D_S is not only nonvoid but also contains sufficiently many members in the following sense: D_S is dense in D with respect to the metric

$$\rho(P_1, P_2)=\left(\int_R |P_1(z)-P_2(z)|\, dxdy\right)^*$$

on D, where $a^*=a/(1+a)$ for nonnegative numbers and $\infty^*=1$. Namely, we shall prove the following

Theorem. *The subspace $D_S(R)$ of singular densities is dense in the metric space $(D(R), \rho)$ for any open Riemann surface R.*

Proof. We only have to show that for any $P\in D$ and any $\eta>0$, there exists a $Q\in D_S$ such that

(2) $$\int_R |P(z)-Q(z)|\, dxdy\leq\eta.$$

Our proof goes on an analogous way to [5]. Let $(\{z_j\}, \{U_j\}, \{\eta_j\})$ $(j=1,2,\cdots)$ be a system such that $\{z_j\}$ is a sequence of points in R not accumulating in R, U_j are parametric disks on R with centers z_j such that $\overline{U}_j\cap\overline{U}_k=\emptyset$ $(j\neq k)$, and $\{\eta_j\}$ is a sequence with $\eta_j>0$ and $\sum_{j=1}^{\infty}\eta_j=\eta$. Furthermore we denote by V_j the concentric parametric disk $|z|<\rho_j=\exp(-4\pi/\eta_j)$ of U_j $(j=1,2,\cdots)$. Let $G(z,\zeta)$ be the Green's function on $S=R-\bigcup_{j=1}^{\infty}\overline{V}_j$ for (1). Fix a point $z_0\in S$ and set

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(3) $$\varepsilon_j = \mathrm{Min}_{\zeta \in \partial V_j} \rho_j \frac{\partial}{\partial n_\zeta} G(z_0, \zeta).$$

Since $\varepsilon_j > 0$ (cf. Itô [1]), by Lemma 1 in [5], there exist densities $P_j(z)dxdy$ on R whose supports are contained in V_j such that

(4) $$|(P_j)_f^{V_j}(z_j)| \leq \varepsilon_j \left| \frac{1}{2\pi} \int_0^{2\pi} f(\rho_j e^{i\theta}) d\theta \right|$$

for every f in $C(\partial V_j)$ and $\int_R P_j(z)dxdy \leq \eta_j$ for each $j=1, 2, \cdots$, where P_f^V is the continuous function on $\bar{V}$ such that $P_f^V | \partial V = f$ and P_f^V is a solution of (1) on V. Using the above densities P_j we define

$$Q(z)dxdy = P(z)dxdy + \sum_{j=1}^\infty P_j(z)dxdy$$

on R. Clearly $Q(z)dxdy$ satisfies the inequality (2). We have to prove that $Q \in D_S$. Let $u(z)$ be a nonnegative solution of $\Delta u(z) = Q(z)u(z)$ on R. As in [5] take a regular exhaustion $\{R_j\}_{j=1}^\infty$ of R such that $z_0 \in R_1$, $R_n \supset \bigcup_{j=1}^n \bar{V}_j$ and $R - \bar{R}_n \supset \bigcup_{j=n+1}^\infty V_j$. Consider a boundary function $u_{n,k}$ $(n<k)$ for the region $S_k = R_k - \bigcup_{j=1}^k \bar{V}_j$ such that $u_{n,k} = u$ on $B_n = \bigcup_{j=1}^n \partial V_j$ and $u_{n,k} = 0$ on $\partial S_k - B_n$. Since $Q(z)dxdy = P(z)dxdy$ on S_k, $u(z)$ is a nonnegative solution of (1) on S_k. Therefore the maximum principle for subharmonic functions yields

(5) $$P_{u_{n,k}}^{S_k}(z_0) \leq u(z_0) \qquad (n=1, 2, \cdots; \ k=n+1, n+2, \cdots).$$

Let $G_k(z, \zeta)$ be the Green's function on S_k for (1). Then by the Green formula

(6) $$P_{u_{n,k}}^{S_k}(z_0) = \frac{1}{2\pi} \sum_{j=1}^n \int_{\partial V_j} u(\zeta) \frac{\partial}{\partial n_\zeta} G_k(z_0, \zeta) ds_\zeta.$$

On letting $k \to \infty$ in (6), we deduce by (3) and (5) that

(7) $$\sum_{j=1}^n \varepsilon_j \int_0^{2\pi} u(\rho_j e^{i\theta}) d\theta \leq 2\pi u(z_0)$$

for every n. On the other hand, since $u(z) = (P_j)_u^{V_j}(z)$ on ∂V_j and $Q \geq P_j$ on $\bar{V}_j$, the comparison principle yields

$$u(z_j) \leq (P_j)_u^{V_j}(z_j) \qquad (j=1, 2, \cdots).$$

By the above inequality with (4), (7), and $u \geq 0$, we have that $\lim_{j\to\infty} u(z_j) = 0$, i.e. $\inf_R u = 0$. Thus we conclude that $Q \in D_S$. Q.E.D.

Remark. Since the density 0 belongs to the ρ-closure of $D_S(R)$, we in particular have

(8) $$D_S(R) \cap L^1(R) \neq \emptyset$$

which is the full content of Nakai [5]. We remark that $L^1(R)$ cannot be replaced by $L^p(R)$ $(1 < p \leq \infty)$ even for the simplest $R = \{|z| < 1\}$.

First observe that if there exist a constant $\delta > 0$ and a compact subset X of a hyperbolic Riemann surface R which is the closure of a regular subregion of R such that

(9) $$\int_W H(z, \zeta) P(\zeta) d\xi d\eta < 2\pi - \delta \qquad (\zeta = \xi + i\eta)$$

for any $z \in W \equiv R - X$, where $H(z, \zeta)$ is the harmonic Green's function on R, then $P \notin D_S$. In fact, the reduction operator $T_p: PB(W) \to HB(W)$ is surjective and then we have

$$e_P(z) = 1 - \frac{1}{2\pi}\int_W H_W(z,\zeta)e_P(\zeta)P(\zeta)d\xi d\eta \tag{10}$$

where e_P is the P-unit on W and $H_W(z,\zeta)$ is the harmonic Green's function on W (cf. Nakai [3], [4]). By (9), (10), $H_W(z,\zeta) \leq H(z,\zeta)$, and $0 < e_P < 1$, we deduce that $e_P(z) \geq \delta/(2\pi)$ for $z \in W$, i.e. $\inf_W e_P > 0$. By the remark in no. 3 in [5], we conclude that (9) implies $P \notin D_S$. We next show that if the density $P \in L^p(R)$ $(1 < p \leq \infty)$, then $P \notin D_S(R)$, where $R = \{z; |z| < 1\}$, i.e. $D_S(R) \cap L^p(R) = \emptyset$ $(1 < p \leq \infty)$. Let $H(z,\zeta)$ be the harmonic Green's function on R and set $R_n = \{z; |z| < 1 - 1/n\}$. Clearly

$$\lim_{n\to\infty}\int_{R-R_n} H(z,\zeta)^q d\xi d\eta = 0 \qquad q = p/(p-1)) \tag{11}$$

uniformly with respect to z. On the other hand, by Hölder's inequality, we have

$$\int_{R-R_n} H(z,\zeta)P(\zeta)d\xi d\eta \leq \left(\int_{R-R_n} H(z,\zeta)^q d\xi d\eta\right)^{1/q}\left(\int_{R-R_n} P(\zeta)^p d\xi d\eta\right)^{1/p}. \tag{12}$$

In view of $P \in L^p(R)$ and (11), the left hand side of (12) satisfies the condition (9) for sufficiently large n. Thus we conclude that $P \notin D_S(R)$.

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