185. An Oscillation Theorem for Differential Equations with Deviating Argument

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In this paper we are concerned with the oscillatory behavior of bounded solutions of the following differential equations with deviating argument

(A) $$[r(t)x^{(n-1)}(t)]' + a(t)f(x(g(t))) = b(t),$$

(B) $$[r(t)x'(t)]^{(n-1)} + a(t)f(x(g(t))) = b(t).$$

For these equations the following conditions are assumed to hold:

(a) $a, b \in C[R_+, R]$, $R_+ = (0, \infty)$, $R = (-\infty, \infty)$;

(b) $r \in C[R_+, R_+]$, and $\int^\infty r^{-1}(t)dt = \infty$;

(c) $f \in C[R, R]$, $yf(y) > 0$ for $y \neq 0$ and $f(y)$ is nondecreasing;

(d) $g \in C[R_+, R_+]$, and $\lim_{t\to\infty} g(t) = \infty$.

We shall consider only those solutions of (A) and (B) which exist on some half-line $[t_0, \infty)$, where t_0 may depend on the particular solution, and are nontrivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

In case $a(t)$ is eventually nonnegative, oscillation and nonoscillation criteria for (A), (B) or related equations have been extensively developed during the past few years. (See e.g. Kusano and Onose [2], [3], Ševelo and Vareh [4], Singh [5].) However, much less is known about the case when $a(t)$ is allowed to oscillate, that is, to take both positive and negative values for arbitrarily large t. The only reference devoted to the latter case is (to our knowledge) the paper by Travis [6] in which it is shown by an example that the well-known Wintner-Leighton oscillation theorem for second order ordinary differential equations cannot directly be extended to the corresponding differential equations with functional argument.

In this paper an attempt will be made to establish an oscillation criterion for bounded solutions of the equations (A) and (B) with oscillating coefficient $a(t)$. Our result is an extension of a theorem of Kartsatos [1, Theorem 1] on the bounded oscillation of second order ordinary differential equations.

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Theorem. *Let conditions* (a)–(d) *hold. Assume that*

$$\int^{\infty} a_+(t)dt=\infty, \qquad \int^{\infty} a_-(t)dt>-\infty, \tag{1}$$

where $a_+(t)=\max\{a(t), 0\}$ *and* $a_-(t)=\min\{a(t), 0\}$, *and*

$$\int^{\infty} |b(t)|\, dt<\infty. \tag{2}$$

Then, every bounded solution $x(t)$ *of* (A) [*or* (B)] *is either oscillatory or such that*

$$\liminf_{t\to\infty} |x(t)|=0. \tag{3}$$

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (A). Without loss of generality we may assume that $x(t)$ is eventually positive. If (3) does not hold, then there are positive numbers m, M and T such that

$$m\leqq x(g(t))\leqq M \qquad \text{for } t\geqq T. \tag{4}$$

Integrating (A) from T to t and taking (c) into account, we obtain

$$\begin{aligned} &r(t)x^{(n-1)}(t)-r(T)x^{(n-1)}(T)\\ &=-\int_T^t a_+(s)f(x(g(s)))ds-\int_T^t a_-(s)f(x(g(s)))ds+\int_T^t b(s)ds\\ &\leqq -f(m)\int_T^t a_+(s)ds-f(M)\int_T^t a_-(s)ds+\int_T^t b(s)ds. \end{aligned} \tag{5}$$

Letting $t\to\infty$ in (5) and using (1) and (2), we have

$$\lim_{t\to\infty} r(t)x^{(n-1)}(t)=-\infty,$$

from which by use of (b) we conclude that

$$\lim_{t\to\infty} x^{(n-1)}(t)=-\infty \quad \text{and hence} \quad \lim_{t\to\infty} x(t)=-\infty.$$

But this contradicts the fact that $x(t)$ is eventually positive. Thus, our assertion for (A) is true.

Next, let $x(t)$ be a bounded solution of (B) such that $\liminf_{t\to\infty} x(t)>0$. Then, (4) holds for some positive m, M and T, and a similar argument as above yields

$$\lim_{t\to\infty} [r(t)x'(t)]^{(n-2)}=-\infty,$$

which again leads to the contradictory conclusion that $\lim_{t\to\infty} x(t)=-\infty$. This proves the assertion for equation (B).

The following example due to Travis [6] shows that our condition (1) cannot be weakened to the following:

$$\int^{\infty} a(t)dt=\infty. \tag{6}$$

Example. The retarded differential equation

$$x''(t)-\frac{\sin t}{2+\sin t}x(t-\pi)=0$$

has a solution $x(t)=2-\sin t$ which is neither oscillatory nor such that $\liminf_{t\to\infty}|x(t)|=0$. The coefficient $a(t)=\sin t/(2+\sin t)$ satisfies (6).

However, $a(t)$ does not satisfy (1):

$$\int^{\infty} a_{+}(t)dt=\infty, \qquad \int^{\infty} a_{-}(t)dt=-\infty.$$

Travis [6] has demonstrated that under condition (6) the derivative $x'(t)$ of every solution $x(t)$ of the equation

$$x''(t)+a(t)x(g(t))=0$$

is oscillatory. We conclude with the statement that this theorem can be extended to the following equation

$$[r(t)x'(t)]'+a(t)f(x(g(t)))=0 \tag{C}$$

to which both (A) and (B) reduce in the special case when $n=2$.

Proposition. *In addition to* (a)–(d) *assume that* $f(y)$ *is differentiable for* $y\neq 0$ *and* $g(t)$ *is differentiable with* $g'(t)\geqq 0$. *Then, if* (6) *holds, the derivative* $x'(t)$ *of any solution* $x(t)$ *of* (C) *is oscillatory.*

We do not give the proof of this proposition, for it is almost a duplication of the proof worked out by Travis.

References

[1] A. G. Kartsatos: Properties of bounded solutions of nonlinear equations of second order. Proc. Amer. Math. Soc., **19**, 1057–1059 (1968).

[2] T. Kusano and H. Onose: Oscillation of solutions of nonlinear differential delay equations of arbitrary order. Hiroshima Math. J., **2**, 1–13 (1972).

[3] ——: Oscillations of functional differential equations with retarded argument. J. Differential Equations, **15**, 269–277 (1974).

[4] V. N. Ševelo and N. V. Vareh: On the oscillation of solutions of the equation $[r(t)y^{(n-1)}(t)]'+p(t)f(y(\tau(t)))=0$. Ukrain. Mat. Ž., **25**, 707–714 (1973) (in Russian).

[5] B. Singh: A necessary and sufficient condition for the oscillation of an even order nonlinear delay differential equation. Canad. J. Math., **25**, 1078–1089 (1973).

[6] C. C. Travis: Oscillation theorems for second-order differential equations with functional arguments. Proc. Amer. Math. Soc., **31**, 199–202 (1972).