35. *On Stationary Point Sets of $(Z_2)^k$-Manifolds*

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1. Definitions. In order to state the results we define some notions.

Let G be a finite group, and $\mathscr{F}$, $\mathscr{F}'$ be families of subgroups of G with $\mathscr{F}\supset\mathscr{F}'$. An $(\mathscr{F}, \mathscr{F}')$-free G-manifold is a pair (M, φ) consisting of a compact differentiable manifold M and a differentiable G-action $\varphi: G\times M\to M$ on M such that

(i) if $x\in M$, then the isotropy group $G_x\in\mathscr{F}$, and

(ii) if $x\in\partial M$, then $G_x\in\mathscr{F}'$.

We may define the unoriented bordism module $\mathfrak{N}_*(G;\mathscr{F},\mathscr{F}')$, over the unoriented cobordism ring $\mathfrak{N}_*$, which consists of bordism classes of $(\mathscr{F},\mathscr{F}')$-free G-manifolds (see Stong [2]). If $\mathscr{F}'$ is empty, we write $\mathfrak{N}_*(G;\mathscr{F})$ for this module.

Let F be the stationary point set of a G-mainfold (M,φ), and $F=\bigcup_i F_i$ be the decomposition by the connected components. Let $(D(\nu_i),\varphi_i)$ be the G-manifold consisting of the normal disc bundle $D(\nu_i)$ of F_i and the G-action φ_i induced by φ. We suppose that any connected component F_i satisfies

$$[D(\nu_i),\varphi_i]=[F_i][D(V_i),\Psi_i]$$

in $\mathfrak{N}_*(G;\mathscr{F}_A,\mathscr{F}_P)$ for some positive dimensional G-representation (V_i,ψ_i), where $\mathscr{F}_A$ (resp., $\mathscr{F}_P$) is the family of all subgroups (resp., all proper subgroups) of G and $D(V_i)$ is the unit disc of V_i. We say in this case that *F has a trivial normal bundle in the weak sense.* When we further suppose that $\dim F_i=\dim F_j$ implies $(V_i,\psi_i)\cong(V_j,\psi_j)$ as G-representations, we say that *F has a trivial normal bundle* (in the sense of Conner-Floyd [1; § 42]).

2. Statement of results. In this note we study the case in which G is $(Z_2)^k$, the direct product of k copies of the multiplicative cyclic group $Z_2=\{1,-1\}$. We obtain the following results:

Theorem 1. *If the stationary point set F of a closed $(Z_2)^k$-manifold (M,φ) has a trivial normal bundle, then we obtain*

(i) $[F]=0$ *in* $\mathfrak{N}_*$, *and*

(ii) $[M,\varphi]=0$ *in* $\mathfrak{N}_*((Z_2)^k;\mathscr{F}_A)$.

Corollary 2 (Conner-Floyd [1: (31.3)]). *The stationary point set F of a positive dimensional closed $(Z_2)^k$-manifold can not consist of one point.*

Theorem 3. *Let F be the stationary point set of a closed $(Z_2)^k$-manifold. If F has a trivial normal bundle in the weak sense, and consists of two connected components F_1, F_2, we obtain*

(i) $[F_1]=[F_2]$ *in* $\mathfrak{N}_*$, *and*

(ii) *if* $[F_1]=[F_2]\neq 0$, *then* F *has a trivial normal bundle.*

3. Preliminaries of proofs. First we state the known results.

Proposition 4. *There exists a short exact sequence*

$$0\longrightarrow\mathfrak{N}_*((Z_2)^k;\mathcal{F}_A)\xrightarrow{j}\mathfrak{N}_*((Z_2)^k;\mathcal{F}_A,\mathcal{F}_P)\xrightarrow{\partial}\mathfrak{N}_*((Z_2)^k;\mathcal{F}_P)\longrightarrow 0,$$

where j is induced by the inclusion $(\mathcal{F}_A,\phi)\subset(\mathcal{F}_A,\mathcal{F}_P)$ and ∂ is induced by restricting a $(Z_2)^k$-action to boundary.

This proposition was obtained in the proof of Proposition 2 in Stong [2].

Let $\mathcal{F}_1$ be the family consisting of only the identity subgroup of Z_2. Then $\mathfrak{N}_*(Z_2;\mathcal{F}_1)$ is the bordism module of free involutions on closed manifolds.

Proposition 5 (Conner-Floyd [1; Theorem 23.2]). *$\mathfrak{N}_*(Z_2:\mathcal{F}_1)$ is the free $\mathfrak{N}_*$-module on generators $\{[S^n,a]\,|\,n=0,1,2,\cdots\}$, where a is the antipodal involution on sphere.*

For a positive integer k let $\mathcal{S}(k)$ be the set of non-empty subsets of $\{1,2,\cdots,k\}$. For any $A\in\mathcal{S}(k)$, let (V_A,φ_A) be the real 1-dimensional $(Z_2)^k$-representation defined by

$$\varphi_A((t_1,\cdots,t_k),v)=(\Pi_{i\in A}t_i)\cdot v$$

for $(t_1,\cdots,t_k)\in(Z_2)^k$ and $v\in V_A$. Then $\{(V_A,\varphi_A)\,|\,A\in\mathcal{S}(k)\}$ gives a complete set of non-trivial irreducible representations of $(Z_2)^k$.

Let Z^+ be the non-negative integers. For any correspondence $f:\mathcal{S}(k)\to Z^+$ we define a $(Z_2)^k$-representation $(V(f),\varphi(f))$ to be the direct sum $\bigoplus_{A\in\mathcal{S}(k)}(V_A,\varphi_A)^{f(A)}$ where $(V_A,\varphi_A)^{f(A)}$ is the direct sum of $f(A)$ copies of (V_A,φ_A). If two correspondences $f,g:\mathcal{S}(k)\to Z^+$ satisfy $f(A)\leqq g(A)$ for all $A\in\mathcal{S}(k)$, we may regard $V(f)$ as a $(Z_2)^k$-subspace of $V(g)$.

We denote the unit dise, the unit sphere of $V(f)$ by $D(f), S(f)$, respectively.

By elementary computations we obtain

Lemma 6. *For any $A\in\mathcal{S}(k)$ there is a subgroup H_A of $(Z_2)^k$ such that*

(i) *H_A is isomorphic to $(Z_2)^{k-1}$, and*

(ii) *for any correspondence $f:\mathcal{S}(k)\to Z^+$ the stationary point set of $(S(f),\varphi(f)|H_A)$ is $S(f(A)\varepsilon_A)$, where $f(A)\varepsilon_A$ is the correspondence defined by*

$$f(A)\varepsilon_A(A')=\begin{cases}f(A) & \text{if } A'=A\\ 0 & \text{if } A'\neq A.\end{cases}$$

Let F be the stationary point set of a closed $(Z_2)^k$-manifold (M,φ), and F_i $(i=1,\cdots,s)$ be the connected components of F. Let $(D(\nu_i),\varphi_i)$

be the $(Z_2)^k$-manifold consisting of the normal disc bundle of F_i and the $(Z_2)^k$-action induced by φ. We suppose that F has a trivial normal bundle in the weak sense. Then

$$[D(\nu_i), \varphi_i] = [F_i][D(f_i), \varphi(f_i)]$$

in $\mathfrak{N}_*((Z_2)^k; \mathcal{F}_A, \mathcal{F}_P)$ for some correspondence f_i.

Lemma 7. *Let $F = \bigcup_{i=1}^s F_i$ be the stationary point set of (M, φ) as in above, then we obtain*

$$\Sigma_{i=1}^s [F_i][S^{f_i(A)-1}, a] = 0$$

in $\mathfrak{N}_*(Z_2; \mathcal{F}_1)$ for all $A \in \mathcal{S}(k)$. (Here we consider $S^{f_i(A)-1}$ to be the empty set for $f_i(A) = 0$.)

Proof. In the exact sequence of Proposition 4

$$\begin{aligned} 0 = \partial j([M, \varphi]) &= \partial(\Sigma_i [D(\nu_i), \varphi_i]) \\ &= \partial(\Sigma_i [F_i][D(f_i), \varphi(f_i)]) \\ &= \Sigma_i [F_i][S(f_i), \varphi(f_i)]. \end{aligned}$$

This means that there exists an $\mathcal{F}_P$-free $(Z_2)^k$-manifold (N, Ψ) such that

$$(\partial N, \psi) = \bigcup_i F_i \times (S(f_i), \varphi(f_i)).$$

For $A \in \mathcal{S}(k)$, let H_A be a subgroup of $(Z_2)^k$ obtained by Lemma 6, and H_A^c be a complement of H_A in $(Z_2)^k$. We denote by N_A the stationary point set of $(N, \Psi | H_A)$. Then, by Lemma 6,

$$(\partial N_A, \Psi | H_A^c) = \bigcup_i F_i \times (S^{f_i(A)-1}, a).$$

Since $(N_A, \Psi | H_A^c)$ is a free Z_2-manifold,

$$\Sigma_i [F_i][S^{f_i(A)-1}, a] = 0 \qquad \text{in } \mathfrak{N}_*(Z_2; \mathcal{F}_1).$$

4. Proof of Theorem 1. (i) For any i with $0 \leqq i < \dim M$, let F^i be the i-dimensional component of F, and $(D(\nu_i), \varphi_i)$ be the $(Z_2)^k$-manifold obtained from the normal disc bundle of F^i. Since F has a trivial normal bundle

$$[D(\nu_i), \varphi_i] = [F^i][D(f_i), \varphi(f_i)]$$

in $\mathfrak{N}_*((Z_2)^k; \mathcal{F}_A, \mathcal{F}_P)$ for some f_i. By Lemma 7

$$\Sigma_i [F^i][S^{f_i(A)-1}, a] = 0$$

in $\mathfrak{N}_*(Z_2; \mathcal{F}_1)$ for all $A \in \mathcal{S}(k)$. Hence, by Proposition 5, $[F^i] = 0$ in $\mathfrak{N}_*$ for i with $f_i(A) \neq 0$. For any i we may appropriately choose A in $\mathcal{S}(k)$ so that $f_i(A) \neq 0$. Thus $[F^i] = 0$ for all i, i.e., $[F] = 0$.

(ii) In the exact sequence of Proposition 4

$$j([M, \varphi]) = \Sigma_i [F^i][D(f_i), \varphi(f_i)] = 0.$$

Since j is monic, $[M, \varphi] = 0$ in $\mathfrak{N}_*((Z_2)^k; \mathcal{F}_A)$.

5. Proof of Theorem 3. (i) Let $(D(\nu_i), \varphi_i)$ be the $(Z_2)^k$-manifold obtained from the normal disc bundle of F_i for $i = 1, 2$. Then, by the assumption,

$$[D(\nu_i), \varphi_i] = [F_i][D(f_i), \varphi(f_i)]$$

in $\mathfrak{N}_*((Z_2)^k; \mathcal{F}_A, \mathcal{F}_P)$ for some f_i. By Lemma 7

$$(*) \qquad \Sigma_i [F_i][S^{f_i(A)-1}, a] = 0$$

in $\mathfrak{N}_*(Z_2; \mathcal{F}_1)$ for any $A \in \mathcal{S}(k)$. From Proposition 5 and the fact that

$f_i(A) \neq 0$ for some $A \in \mathcal{S}(k)$ we obtain (i).

(ii) It is sufficient (and necessary) that $f_1 = f_2$. If $f_1(A) \neq f_2(A)$ for some $A \in \mathcal{S}(k)$ and $f_1(A) \neq 0$, then $[F_1] = 0$ from the equation (*). This is a contradiction.

References

[1] P. E. Conner and E. E. Floyd: Differentiable Periodic Maps. Springer-Verlag (1964).
[2] R. E. Stong: Equivariant bordism and $(Z_2)^k$ actions. Duke Math. J., **37**, 779–785 (1970).