70. *On the Structure of the Graded C-Algebras of Theta Functions*

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Introduction. Let $(X, \mathcal{L})$ be a polarized abelian variety of dimension n over the complex number field $\boldsymbol{C}$ defined by an abelian variety X and an ample invertible sheaf $\mathcal{L}$ on X. The purpose of this note is to give some information on the graded $\boldsymbol{C}$-algebra $\mathcal{A}(X, \mathcal{L}) = \bigoplus_{\alpha=0}^{\infty} H^0(X, \mathcal{L}^\alpha)$. We understand, by the Siegel space $\mathcal{H}_n$ of degree n, the set of all complex symmetric matrices with positive imaginary part. We have, for $z \in \mathcal{H}_n$ and a square matrix e of size n with coefficients in $\boldsymbol{Z}$ and $\det e \neq 0$, a naturally polarized complex torus $\boldsymbol{C}^n/\langle z, e\rangle$ where $\langle z, e\rangle$ is the lattice subgroup of $\boldsymbol{C}^n$ generated by the column vectors of the $(n\times 2n)$-matrix (z, e). After a suitable choice of (z, e), $\boldsymbol{C}^n/\langle z, e\rangle$ is isomorphic to $(X, \mathcal{L})$ and the investigation of $\mathcal{A}(X, \mathcal{L})$ is reduced to that of the graded $\boldsymbol{C}$-algebra consisting of theta functions on $\boldsymbol{C}^n$ relative to $\langle z, e\rangle$ of certain types.

Besides the above notations $\mathcal{H}_n$, e, etc., we indicate some definitions and notations which are used throughout this note. $\boldsymbol{C}^n$ etc. denote the set of column n-vectors with components in $\boldsymbol{C}$ etc. $\boldsymbol{Z}_+$ and $\boldsymbol{N}$ denote the set of all positive integers and $\boldsymbol{Z}_+ \cup \{0\}$, respectively. For $\alpha, \beta \in \boldsymbol{Z}_+$, g.c.d. (α, β) is the greatest common divisor of α and β. In general for e as above (resp. $\alpha \in \boldsymbol{Z}_+$), U_e (resp. U_α) denotes a complete set of representatives of ${}^te^{-1}\boldsymbol{Z}^n$ (resp. $\alpha^{-1}\boldsymbol{Z}^n$) mod $\boldsymbol{Z}^n$, which is once for all fixed through in a discussion. When we write $k=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \boldsymbol{R}^{2n}$, k_1 and k_2 are its upper and lower halves in $\boldsymbol{R}^n$. We put $\boldsymbol{e}(t)=\exp(2\pi\sqrt{-1}t)$. $\theta[k](z\,|\,x)$ is a holomorphic function on $(z, x) \in \mathcal{H}_n \times \boldsymbol{C}^n$, which is defined by

$$\theta[k](z\,|\,x)=\sum_{r\in \boldsymbol{Z}^n} \boldsymbol{e}\Big\{\frac{1}{2}{}^t(r+k_1)z(r+k_1)+{}^t(r+k_1)(x+k_2)\Big\}.$$

For $\alpha \in \boldsymbol{N}$, $z \in \mathcal{H}_n$ and $m=\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \boldsymbol{R}^{2n}$, an entire function $\xi(x)$ not identically zero on $\boldsymbol{C}^n$ is called a theta function of type $((z, e), m)_\alpha$, if the period relation, for any $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \boldsymbol{Z}^{2n}$,

$$\xi\Big(x+(z, e)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\Big)\cdot \xi(x)^{-1}=\boldsymbol{e}\Big\{\alpha\Big(-{}^ts_1x-\frac{1}{2}{}^ts_1zs_1+({}^ts_1, {}^ts_2)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\Big)\Big\}$$

holds. $\Theta_\alpha((z,e),m)$ is the totality of theta functions of type $((z,e),m)_\alpha$ plus $\{0\}$. We know that $\Theta_0((z,e),m)=\boldsymbol{C}$ and $\Theta_\alpha((z,e),m)$ is a $\boldsymbol{C}$-vector space of dimension $\alpha^n\,|\det e|$ if $\alpha\in\boldsymbol{Z}_+$. The graded $\boldsymbol{C}$-algebra $\mathcal{A}((z,e),m)$ of theta functions of type $((z,e),m)$ is defined to be $\bigoplus_{\alpha=0}^{\infty}\Theta_\alpha((z,e),m)$, and integrally closed in its field of fractions.

Precise details of this note will appear in American Journal of Mathematics [2].

1. The problems. Taking investigations on the structure of the graded $\boldsymbol{C}$-algebra $\mathcal{A}((z,e),m)$ into consideration, (which is isomorphic to $\mathcal{A}(X,\mathcal{L})$ if we choose suitable $((z,e),m)$,) we concern ourselves with the following three questions:

(a) determine a canonical $\boldsymbol{C}$-linear base $\{\theta_1^{(\alpha)},\cdots,\theta_{\alpha^n|\det e|}^{(\alpha)}\}$ of $\Theta_\alpha((z,e),m)$ for each $\alpha\in\boldsymbol{N}$;

(b) in the multiplication map, for α and β in $\boldsymbol{N}$,

$$\Theta_\alpha((z,e),m)\times\Theta_\beta((z,e),m)\to\Theta_{\alpha+\beta}((z,e),m),$$

make the multiplication table among elements of bases such that

$$\theta_i^{(\alpha)}\cdot\theta_j^{(\beta)}=\sum_{k=1}^{(\alpha+\beta)^n\cdot|\det e|}c_{ij,k}\theta_k^{(\alpha+\beta)},\qquad 1\leq i\leq\alpha^n\cdot|\det e|,\ 1\leq j\leq\beta^n\cdot|\det e|;$$

(c) find out a set of generators of the $\boldsymbol{C}$-algebra $\mathcal{A}((z,e),m)$ over $\boldsymbol{C}$.

The following well-known proposition answers the first question (a).

Proposition 1 (The base theorem). *For $z\in\mathfrak{H}_n$, $k=\begin{pmatrix}k_1\\k_2\end{pmatrix}\in\boldsymbol{R}^{2n}$ and $\alpha\in\boldsymbol{Z}_+$, we have*

(i) *$\theta[k](\alpha z\,|\,\alpha x)$ is a theta function of type $\left((z,e),\begin{pmatrix}-\alpha^{-1}k_2\\{}^tek_1\end{pmatrix}\right)_\alpha$*

(ii) *$\left\{\theta\begin{bmatrix}k_1+p_1\\\alpha k_2\end{bmatrix}(\alpha z\,|\,\alpha x)\,|\,p_1\in U_{\alpha e}\right\}$ is a $\boldsymbol{C}$-base of $\Theta_\alpha\left((z,e),\begin{pmatrix}-k_2\\{}^tek_1\end{pmatrix}\right)$, (which will be treated as canonical base in this note).*

2. Generalized addition formulas. The following formula which is proved just by means of computation of the series, is fundamental in the approach to all our process.

Theorem 2 (Generalized addition formulas). *For $k^{(1)},k^{(2)}\in\boldsymbol{R}^{2n}$, and $\alpha,\beta,\gamma\in\boldsymbol{Z}_+$ such that α is divisible by γ, we have*

(2.a) $\theta[k^{(1)}](\alpha z\,|\,\alpha x^{(1)})\cdot\theta[k^{(2)}](\beta z\,|\,\beta x^{(2)})$

$$=\sum_{p_1\in U_{\alpha+\beta}}\left\{\sum_{c_1\in U_\gamma}\theta\begin{bmatrix}\gamma^{-1}(\alpha+\beta)^{-1}(\alpha k_1^{(1)}+\beta k_1^{(2)})+\gamma^{-1}\alpha p_1+c_1\\\gamma(k_2^{(1)}+k_2^{(2)})\end{bmatrix}(\gamma^2(\alpha+\beta)z\,|\,\gamma(\alpha x^{(1)}+\beta x^{(2)}))\right\}$$

$$\times\theta\begin{bmatrix}(\alpha+\beta)^{-1}(-k_1^{(1)}+k_1^{(2)})-p_1\\-\beta k_2^{(1)}+\alpha k_2^{(2)}\end{bmatrix}(\alpha\beta(\alpha+\beta)z\,|\,\alpha\beta(-x^{(1)}+x^{(2)})).$$

The following (2.b) follows easily from (2.a) by simple substitutions (especially $\gamma=1$ etc.).

Corollary 2.1. *For* $k\in \boldsymbol{R}^{2n}$; $\alpha,\beta\in \boldsymbol{Z}_+$; *and* $(a_1,b_1)\in U_{\alpha e}\times U_{\beta e}$, *we have*

$$\text{(2.b)}\quad \theta\begin{bmatrix}k_1+a_1\\ \alpha k_2\end{bmatrix}(\alpha z\,|\,\alpha x)\cdot\theta\begin{bmatrix}k_1+b_1\\ \beta k_2\end{bmatrix}(\beta z\,|\,\beta x)$$
$$=\sum_{p_1\in U_{\alpha+\beta}}\theta\begin{bmatrix}k_1+(\alpha+\beta)^{-1}(\alpha a_1+\beta b_1)+\alpha p_1\\ (\alpha+\beta)k_2\end{bmatrix}((\alpha+\beta)z\,|\,(\alpha+\beta)x)$$
$$\times\theta\begin{bmatrix}(\alpha+\beta)^{-1}(-a_1+b_1)-p_1\\ 0\end{bmatrix}(\alpha\beta(\alpha+\beta)z\,|\,0).$$

We content ourselves with stating this formula (2.b) as an answer to the question (b) at this moment.

3. The rank theorem. Connecting with the problem (c), we consider the question:

(c′) when is the multiplication map

$$\Theta_\alpha((z,e),m)\otimes\Theta_\beta((z,e),m)\to\Theta_{\alpha+\beta}((z,e),m)$$

surjective?

From the formula (2.b) we can image that some result concerning the rank of a certain matrix with theta constants as its coefficients is needed. That is the reason why we give the following theorem, which is essential in the sequel.

Theorem 3 (The rank theorem). *For* $\alpha,\beta\in Z_+$ *such that* g.c.d. $(\alpha,\beta)=1$ *and* $\beta>\alpha$, $l\in \boldsymbol{R}^{2n}$, $z^{(0)}\in\mathcal{H}_n$ *and* $x^{(0)}\in \boldsymbol{C}^n$, *we have*

$$\operatorname{rank}\left(\theta\left[l+\begin{pmatrix}b_1+a_1\\ 0\end{pmatrix}\right](z^{(0)}\,|\,x^{(0)})\right)_{(b_1,a_1)\in U_\beta\times U_\alpha}=\alpha^n.$$

For proof of the theorem we use the formula (3.a) for $a_2\in U_\alpha$, which follows from (2.a) by certain substitutions (esp. $\gamma=\alpha$ etc.):

$$\text{(3.a)}\quad \theta[k^{(1)}](\alpha z\,|\,\alpha x^{(1)})\cdot\theta\left[k^{(2)}+\begin{pmatrix}0\\ (\alpha+\beta)a_2\end{pmatrix}\right](\beta z\,|\,\beta x^{(2)})$$
$$=\boldsymbol{e}((\alpha+\beta)^t k_1^{(2)}a_2)\sum_{p_1\in U_{\alpha+\beta}}\left\{\sum_{a_1\in U_\alpha}\boldsymbol{e}(\alpha(\alpha+\beta)^t a_1a_2)\right.$$
$$\times\theta\begin{bmatrix}\alpha^{-1}(\alpha+\beta)^{-1}(\alpha k_1^{(1)}+\beta k_1^{(2)})+p_1+a_1\\ \alpha(k_2^{(1)}+k_2^{(2)})\end{bmatrix}$$
$$\left.(\alpha^2(\alpha+\beta)z\,|\,\alpha(\alpha x^{(1)}+\beta x^{(2)}))\right\}$$
$$\times\theta\begin{bmatrix}(\alpha+\beta)^{-1}(-k_1^{(1)}+k_1^{(2)})-p_1\\ -\beta k_2^{(1)}+\alpha k_2^{(2)}\end{bmatrix}(\alpha\beta(\alpha+\beta)z\,|\,\alpha\beta(-x^{(1)}+x^{(2)}))$$

and the fact that $\left\{\theta\left[k+\begin{pmatrix}0\\ a_2\end{pmatrix}\right](z\,|\,x)\,|\,a_2\in U_\alpha\right\}$ are linearly independent over $\boldsymbol{C}$ (as functions of $x\in \boldsymbol{C}^n$).

4. Two lemmas and the surjection theorem. With the purpose of learning the question (c′), we shall now state two lemmas with rough ideas of proving them, the first of which is related to the principal case (i.e., $e=1_n$) of the question and the second of which works as a bridge between the principal and the general cases.

Lemma 4.1. *Let $k^{(1)}, k^{(2)}$ be two vectors in $\boldsymbol{R}^{2n}$ and let z be a point on $\mathcal{H}_n$. If α, β and δ are three positive integers such that $\alpha+\beta<\alpha\beta\delta$ and* g.c.d. $(\alpha+\beta, \alpha\delta)=1$, *then we have that* $\left\{\theta\begin{bmatrix}k_1^{(1)}-\beta c_1\\ \alpha k_2^{(1)}\end{bmatrix}(\alpha z\,|\,\alpha x)\times\theta\begin{bmatrix}k_1^{(2)}+\alpha c_1\\ \beta k_2^{(2)}\end{bmatrix}(\beta z\,|\,\beta x)\,|\,c_1\in U_{\alpha\beta\delta}\right\}$ *spans the space* $\Theta_{\alpha+\beta}\left((z,1_n), (\alpha+\beta)^{-1}\times\begin{pmatrix}-(\alpha k_2^{(1)}+\beta k_2^{(2)})\\ \alpha k_1^{(1)}+\beta k_1^{(2)}\end{pmatrix}\right)$.

This follows from the rank theorem in **3** and from the formula (4.1.a) obtained from (2.a) by certain substitutions (esp. $\gamma=1$ etc.): for $c_1\in U_{\alpha\beta\delta}$,

$$\theta\begin{bmatrix}k_1^{(1)}-\beta c_1\\ \alpha k_2^{(1)}\end{bmatrix}(\alpha z\,|\,\alpha x)\cdot\theta\begin{bmatrix}k_1^{(2)}+\alpha c_1\\ \beta k_2^{(2)}\end{bmatrix}(\beta z\,|\,\beta x)$$
$$=\sum_{p_1\in U_{\alpha+\beta}}\theta\begin{bmatrix}(\alpha+\beta)^{-1}(\alpha k_1^{(1)}+\beta k_1^{(2)})+\alpha p_1\\ \alpha k_2^{(1)}+\beta k_2^{(2)}\end{bmatrix}((\alpha+\beta)z\,|\,(\alpha+\beta)x)$$
$$\times\theta\begin{bmatrix}(\alpha+\beta)^{-1}(-k_1^{(1)}+k_1^{(2)})+c_1-p_1\\ \alpha\beta(-k_2^{(1)}+k_2^{(2)})\end{bmatrix}(\alpha\beta(\alpha+\beta)z\,|\,0). \tag{4.1.a}$$

Lemma 4.2. *We have, for $\alpha\in\boldsymbol{Z}_+, z\in\mathcal{H}_n, k=\begin{pmatrix}k_1\\ k_2\end{pmatrix}\in\boldsymbol{R}^{2n}$ and a matrix e as before,*

$$\Theta_\alpha\left((z,e),\begin{pmatrix}-k_2\\ {}^tek_1\end{pmatrix}\right)=\bigoplus_{p_1\in U_e}\Theta_\alpha\left((z,1_n),\begin{pmatrix}-k_2\\ k_1+\alpha^{-1}p_1\end{pmatrix}\right). \tag{4.2.a}$$

This follows directly from Proposition 1 (ii).

Theorem 4.3 (The surjection theorem). *For $m^{(1)}, m^{(2)}\in\boldsymbol{R}^{2n}$; α, β and $\delta\in\boldsymbol{Z}_+$ such that $\alpha+\beta<\alpha\beta\delta$ and* g.c.d. $(\alpha+\beta,\alpha\delta)=1$; *we have*

$$\Theta_{\alpha+\beta}((z,e),(\alpha+\beta)^{-1}(\alpha m^{(1)}+\beta m^{(2)}))$$
$$=\sum_{d_1\in U_\delta}\Theta_\alpha\left((z,e),m^{(1)}-\begin{pmatrix}0\\ \alpha^{-1\,t}ed_1\end{pmatrix}\right)\cdot\Theta_\beta\left((z,e),m^{(2)}+\begin{pmatrix}0\\ \beta^{-1\,t}ed_1\end{pmatrix}\right). \tag{4.3.a}$$

The principal case (i.e., $e=1_n$) of this theorem follows from Lemma 4.1 and the general case is proved by combining the principal case and Lemma 4.2.

5. Main results. We conclude this note with a theorem and two corollaries which answer the questions (c′) and (c), and also furnish information concerning projective normality of projective models of abelian varieties in order of mention.

Theorem 5.1. *For $\alpha,\beta\in\boldsymbol{Z}_+$ such that $\alpha\geq 2$, $\beta\geq 3$, and $m\in R^{2n}$, we have*

$$\Theta_{\alpha+\beta}((z,e),m)=\Theta_\alpha((z,e),m)\cdot\Theta_\beta((z,e),m). \tag{5.1.a}$$

The equality $\Theta_5((z,e),m)=\Theta_2((z,e),m)\cdot\Theta_3((z,e),m)$ follows from (4.3.a) by putting $\alpha=2$. $\beta=3$ and $\delta=1$, and the general case follows from the same (4.3.a) by induction.

Corollary 5.2. *The graded $\boldsymbol{C}$-algebra $\mathcal{A}((z, e), m)$ is generated by* $\bigcup_{1 \leq \alpha \leq 4} \Theta_\alpha((z, e), m)$ *over* $\boldsymbol{C}$.

Corollary 5.3. *If* $e \equiv 0 \pmod{\gamma}$ *for* $\gamma \geq 3$, *the graded* $\boldsymbol{C}$-*algebra* $\mathcal{A}((z, e), m)$ *is generated by* $\Theta_1((z, e), m)$ *over* $\boldsymbol{C}$.

Both corollaries are direct consequences of Theorem 5.1.

References

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