65. On Some Evolution Equations of Subdifferential Operators

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(Comm. by Kôsaku YOSIDA, M. J. A., May 9, 1975)

1. Introduction. In this paper we are concerned with nonlinear evolution equations of a form

$$\frac{du}{dt}+\partial\psi^t u(t)+A(t)u(t)\ni f(t), \qquad 0\leqq t\leqq T \tag{1.1}$$

in a real Hilbert space H. Here for each fixed t, $\partial\psi^t$ is subdifferential of a lower semicontinuous convex function ψ^t from H into $(-\infty,\infty]$, $\psi^t\not\equiv\infty$ and $A(t)$ is a monotone, single valued and hemicontinuous operator which is perturbation in a sense. The effective domain of ψ^t defined by $\{u\in H:\psi^t(u)<+\infty\}=D$ is independent of t. We denote the inner product and the norm in H by (,) and $\|\ \|$ respectively. Let T be a positive constant.

We assume the following conditions for ψ^t and $A(t)$.

A–(1). For every $r>0$ there exists a positive constant $L_1(r)$ such that

$$|\psi^t(u)-\psi^s(u)|\leqq L_1(r)\,|h(t)-h(s)|\,\{\psi^t(u)+1\}$$

hold if $0\leqq s,\ t\leqq T$, $u\in D$ and $\|u\|\leqq r$, where $h(t)$ is a continuous function with bounded total variation.

A–(2). If $u(t)\in D$ is absolutely continuous on $[a,b]$ $(0\leqq a<b\leqq T)$ then $A(t)u(t)$ is strongly measurable on $[a,b]$ and for any fixed $t_0\in[a,b]$ $A(t_0)u(t)$ is also strongly measurable on $[a,b]$. For any fixed $u\in D$, $A(t)u$ is continuous on $[0,T]$.

A–(3). There are Riemann integrable functions $W_r(t)^2$ on $[0,T]$ and a constant $0<K_r<1/2$ such that

$$\|A(t)u\|\leqq K_r\,|||\partial\psi^t u|||+W_r(t) \qquad \text{for any } \|u\|\leqq r.$$

A–(4). If $u(t)$ is absolutely continuous and $|\psi^t(u)|+\|u(t)\|\leqq r$, then $A(t)u(t)\leq W_r(t)^2$.

Under the above assumptions we consider the uniqueness and existence of the solution of (1–1) where the solution is defined as follows:

Definition 1.1. We say that $u(t)$ is a solution of (1–1) if and only if $u(t)$ is continuous on $[0,T]$ and absolutely continuous on $(0,T]$ and if (1–1) holds almost everywhere on $[0,T]$.

Theorem 1.1. *Suppose that the assumptions stated above are satisfied. Then we hold the unique solution of* (1–1) *where* $f\in L_2[0,T;H]$ *and the initial date* $u_0\in\bar{D}$.

Remark 1.1. The continuity assumption A–(1) is weaker than those of J. Watanabe [3] and H. Attouch and A. Damlamian [1].

2. The outline of the proof. Using $\psi^0(a) \geqq C' \|a\| + D'$ and A–(1), we get the following lemma.

Lemma 2.1. *There exist constants C_1 and C_2 which are independent of t and α such that*

$$\psi^t(\alpha) \geq C_1 \|\alpha\| + C_2 \qquad \textit{for any } \alpha \in H.$$

We take a sequence $\{t_i\}_{i=1}^n$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i \in I$ for any $i = 0, 2, \cdots, n$ and $|t_i - t_{i-1}| \to 0$ as $n \to \infty$ for any $i = 1, 2, \cdots, n$. We denote by

$$\psi_n^t(u) = \psi^{t_i}(u),\ A_n(t) = A(t_i),\ \text{for } t_i \leq t < t_{i+1}.$$

We consider the following evolution equations

$$\begin{cases} \dfrac{d}{dt} u_n^i + (\partial \psi_n^t + A_n(t)) u_n^i(t) \ni f(t) & t_i \leq t < t_{i+1} \\ u_n^i(t_i) = u_n^{i-1}(t_i) \quad \text{and} \quad u_n^0(0) = u_0 \in D \quad \text{for } i = 0, 1, \\ \cdots n-1 \quad \text{and} \quad f(t) \in L^2[0, T : H]. \end{cases} \tag{2–1}$$

The solution of (2–1) is defined inductively by the solution of a operator with constant coefficients. For the sake of simplicity we write $u_n(t) = u_n^i(t)$.

Using that $\{u_n(t)\}$ are the solutions of (2–1) and Lemma 1 we get the following lemma.

Lemma 2.2. *There is a constant γ independent of n and t such that*

$$\|u_n(t)\| \leqq \gamma.$$

On the other hand since we get

$$\frac{d}{dt} \psi_n^t(u_n) + \left\| \frac{d}{dt} u_n \right\|^2 = \left(f(t) - A_n(t) u_n, \frac{d}{dt} u_n \right) \qquad \text{a.e.t.}$$

from H. Brezis [2]. Since $u_n(t)$ is a strong solution of (2–1) we see

$$\psi_n^t(u_n(t)) + \delta \int_{t_i}^t \left\| \frac{d}{dt} u_n \right\|^2 dt \leqq \psi_n^{t_i}(u_n(t_i)) + \int_{t_i}^t C_\delta (\|f\| + W_r)^2 ds \tag{2–2}$$

from our assumption A–(3) where δ and C_δ are positive constants independent of n, t and t_i. Combining (2–2) and A–(1) we see

$$\begin{aligned} \psi_n^{t_i}(u_n(t_{i+1})) &\leqq \psi_n^{t_i}(u_n(t_i))\{1 + L_1(\gamma) |h(t_{i-1}) - h(t_i)|\} \\ &\quad + \int_{t_i}^{t_{i+1}} C_\delta (f(s) + W(t_i))^2 ds + L_1(\gamma) |h(t_{i-1}) - h(t_i)|. \end{aligned} \tag{2–3}$$

We put

$$K = \left\{ \int_0^T 2C_\delta \|f\|^2\, ds + 2 \int_0^T w_\gamma^2(t) dt + L_1(\gamma) V(h) + |\psi^0(u_0)| + 1 \right\}$$

then from (2–3) we see

$$|\psi_n^t(u_n(t))| < 3K e^{K L_1(\gamma) V(h)} \tag{2–4}$$

where $V(h)$ = tolal variation of h on $[0, T]$.

Combining (2–3) and (2–4) we get the following lemma.

Lemma 2–3. *We know*

$$|\psi_n^t(u_n(t))|+\int_0^t\left\|\frac{du_n}{dt}\right\|^2 dt\leq C_3$$

where C_3 is a constant independent of n and t.

From the above lemma we know that there exists subsequence $\left\{\frac{d}{dt}u_{n_j}\right\}$ which is L_2-weakly convergent. For the sake of simplicity we put $u_n=u_{n_j}$. Thus we see that $u_n(t)$ is weak convergence to $u(t)$ and $u(t)$ is absolutely continuous on $[0, T]$. On the other hand since $u_n(t)$ is the solution of (2–1) we find

$$\int_0^T \psi_n^s(v(s))ds-\int_0^T \psi_n^s(u_n(s))ds$$
$$\geqq\int_0^T (f(s)-A_n(s)u_n(s)-\frac{d}{ds}u_n(s), v(s)-u_n(s))ds$$
$$\geqq\int_0^T \left(f(s)-A_n(s)v(s)-\frac{d}{ds}v(s), v(s)-u_n(s)\right)ds+1/2\,\|u_0-v(0)\|^2.$$

Then

$$\int_0^T (\psi^s(v(s))-\psi^s(u(s)))ds$$
$$\geqq\int_0^T \left(f(s)-A(s)v(s)-\frac{d}{dt}v(s), v(s)-u(s)\right)ds+1/2\,\|u_0+v(0)\|^2.$$

Next we put $v(t)=pu(t)+(1-p)w(t)$ where $w(t)\in D$ and is absolutely continuous.

Thus we obtain the following inequality

$$\int_0^T (\psi^s(w(s))-\psi^s(u(s)))ds$$
$$\geqq\int_0^T \left(f(s)-A(s)u(s)-\frac{d}{dt}u(s), w(s)-u(s)\right)ds.$$

Next for any fixed $\xi\in D$ and $0\leqq t_1<t_2\leqq T$ we put

$$w(t)=\begin{cases} \xi: & t_1+\varepsilon\leqq t\leqq t_2-\varepsilon \\ pu(t_1)+q\xi: & t=pt_1+q(t_1+\varepsilon) \\ u(t): & 0\leqq t\leqq t_1,\ t_2\leq t\leq T \\ pu(t_2)+q\xi: & t=pt_2+(t_2-\varepsilon)q \end{cases}$$

where $p+q=1$ $p>0, q>0$ and $\varepsilon>0$.

If $\varepsilon\to 0$ we get

$$\int_{t_1}^{t_2}\psi^t(\xi)dt-\int_{t_1}^{t_2}\psi^t(u(t))dt\geqq\int_{t_1}^{t_2}\left(f(t)-A(t)u(t)-\frac{d}{dt}u(t), \xi-u(t)\right)dt.$$

For any Lebesque points of $\psi^t u(t), f(t)$ $A(t)u(t), \frac{d}{dt}u(t)$, and $u(t)$ we know

$$\psi^t(\xi)-\psi^t u(t)\geqq\left(f(t)-A(t)u(t)-\frac{d}{dt}u(t), \xi-u(t)\right).$$

Considering that $\partial\psi^t = A(t)$ is monotone operator we can show the uniqueness of (1–1). If $u_0 \in D$ we can prove the theorem.

Next if $u_0 \in \bar{D}$ we put $u_{m,0} = (1 + 1/m\partial\psi^0)^{-1}u_0$. We denote by $u_m(t)$ the solution of (1–1) of initial data $u_{m,0}$. Since $\partial\psi^t + A(t)$ is monotone operator we see that $u_m(t)$ is uniformly convergent on $[0, T]$ then $\lim_{m\to\infty} u_m(t) = u(t)$.

Using that $u_m(t)$ are strong solutions of (1–1) and A–(3) we know for any $0 < \delta < T$,

$$\int_0^\delta \psi^t(u_m(t))dt \leq C_4$$

where C_4 is a constant independent of δ and m. There exist $0 < \delta_m < \delta$ $m = 1, 2, \cdots$ such that

$$\psi^{\delta_m}(u_m(\delta_m)) \leq \frac{1}{\delta}\int_0^\delta \psi^t(u_m(t))dt \leq \frac{C_4}{\delta} = C_5.$$

We denote by $v_m(t)$ the solution of (1–1) for the initial date $v(\delta_m) = u_m(\delta_m) \in D$ on $[\delta_m, T]$. Then we find $v_m(t) = u_m(t)$ on $[\delta_m, T]$ from the uniqueness of the solution of (1–1). On the other hand noting the method of Lemma 2–3 we get

$$|\psi_n^{t_n}(v_m^n(t))| \leq C_6 \qquad \text{for } t \in [\delta_m, T]$$

where C_6 is independent of n and m.

Thus we get

$$\int_\delta^T \left\|\frac{du_m}{dt}(t)\right\|^2 dt \leq \int_{\delta_m}^T \left\|\frac{dv_m}{dt}(t)\right\|^2 dt \leq C_7.$$

Using the above same method on $[\delta, T]$ we can prove the Theorem.

References

[1] H. Attouch et Damlamian: Problémes dévolution dans Les Hilbert et applications (to appear).

[2] H. Brezis: Propriétés régularisantes de certains semi groupes non linéaires. Israel. J. Math., **9**, 513–534 (1971).

[3] J. Watanabe: On certain nonlinear evolution equations. J. Math. Soc. Japan, **25**, 446–463 (1973).