80. *An H-Theorem for a System of Competing Species*

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§ 1. **Introduction.** Many authors considered the mathematical theory of struggle for existence, after the work of Volterra [7] and Lotka. Kimura gave an interesting example in genetics [5]. We have considered models based on binary collisions, which leads to a Lotka-Volterra equation, in [3] and [4]. By the method used in [3], we can naturally derive equations in case of the system of higher densities, and can prove a theorem. We take into account the mutual interaction of neighbouring three particles. E. G. D. Cohen [1], J. V. Sengers [6] and J. R. Dorfmann [2] extend the Boltzmann equation to the gas of higher densities. Considering their results we develop the model of competing species for the system of higher densities.

§ 2. **Collision algebra.** At first we consider a simple model which satisfies the following:

1) There are three species 1, 2 and 3 whose numbers of particles are at time $t, n_1(t), n_2(t)$ and $n_3(t)$ respectively, where $n_1(t)+n_2(t)+n_3(t)=n$ and n is a constant.

2) Each particle collides with another particle dt times on the average per time length dt.

3) Each particle is in a chaotic bath of particles. Each colliding pair is equally likely chosen.

4) A particle of species i and a particle of species j collide with each other and become two particles of species i, if $i-j\equiv 0, 1 \pmod 3$. If $i-j\equiv 2 \pmod 3$ they become two particles of species j.

Consider the case which n is sufficiently large, we can derive an equation by the following way. Each of $\left(\frac{n_1(t)}{n}dt\right)n_1(t)$ particles of species 1 collides with a particle of species 1 and remains in species 1. Each of $\left(\frac{n_3(t)}{n}dt\right)n_1(t)$ particles of species 1 collides with a particle of species 3 and changes to species 1. Each of $\left(\frac{n_1(t)}{n}dt\right)n_3(t)$ particles of species 3 collides with a particle of species 1 and changes to species 1. So we have

$$dn_1(t)=n_1(t+dt)-n_1(t)$$

$$(1) \qquad =\left\{\frac{n_1(t)}{n}n_1(t)+\frac{n_3(t)}{n}n_1(t)+\frac{n_1(t)}{n}n_3(t)-n_1(t)\right\}dt.$$

In the same way we derive the equations for $n_1(t)$ and $n_2(t)$. So we have

$$(2) \qquad \begin{aligned}\frac{\partial}{\partial t}n_1(t)&=\frac{n_1(t)}{n}n_1(t)+\frac{n_3(t)}{n}n_1(t)+\frac{n_1(t)}{n}n_3(t)-n_1(t)\\ \frac{\partial}{\partial t}n_2(t)&=\frac{n_2(t)}{n}n_2(t)+\frac{n_1(t)}{n}n_2(t)+\frac{n_2(t)}{n}n_1(t)-n_2(t)\\ \frac{\partial}{\partial t}n_3(t)&=\frac{n_3(t)}{n}n_3(t)+\frac{n_2(t)}{n}n_3(t)+\frac{n_3(t)}{n}n_2(t)-n_3(t).\end{aligned}$$

Putting $\left(\frac{n_1(t)}{n},\frac{n_2(t)}{n},\frac{n_3(t)}{n}\right)\equiv(P_1(t),P_2(t),P_3(t))$, (2) is expressed as

$$(3) \qquad \begin{aligned}\frac{\partial}{\partial t}P_1(t)&=P_1^2(t)+P_3(t)P_1(t)+P_1(t)P_3(t)-P_1(t)\\ \frac{\partial}{\partial t}P_2(t)&=P_2^2(t)+P_1(t)P_2(t)+P_2(t)P_1(t)-P_2(t)\\ \frac{\partial}{\partial t}P_3(t)&=P_3^2(t)+P_2(t)P_3(t)+P_3(t)P_2(t)-P_3(t).\end{aligned}$$

Now we define collision algebra.

Definition. Collision algebra S^{2s+1} is defined by the following I, II and III.

I S^{2s+1} is a linear space with the bases $E_1, E_2, \cdots, E_{2s+1}$, more precisely,

i) given $x, y \in S^{2s+1}$ where $x=\sum_{i=1}^{2s+1} x_iE_i$ and $y=\sum_{i=1}^{2s+1} y_iE_i$

$x=y$, iff $x_i=y_i$ for all i,

ii) $x+y=\sum_{i=1}^{2s+1}(x_i+y_i)E_i$

iii) $\lambda x=\sum_{i=1}^{2s+1}\lambda x_iE_i$.

II The product of bases is defined as the following

i) $E_i\circ E_j=E_j\circ E_i$ (commutativity)

ii) $E_i\circ E_j=\begin{array}{l}E_j \text{ iff } i-j\equiv 0, -1, -2, \cdots, -s \pmod{2s+1}\\ E_i \text{ iff } i-j\equiv 0, 1, 2, \cdots, s \pmod{2s+1}.\end{array}$

III The product of two elements $x, y \in S^{2s+1}$ is defined as,

$$\sum_{i=1}^{2s+1} x_iE_i\circ\sum_{j=1}^{2s+1} y_jE_j\equiv\sum_{i,j=1}^{2s+1} x_iy_jE_i\circ E_j$$

which belongs to S^{2s+1}.

In the following line we denote x_i as E_i's component of $x\in S^{2s+1}$.

The collision algebra has the following properties.

a) The multiplication and the addition satisfy the distribution law and the commutative law, that is, for $x, y, z\in S^{2s+1}$

$$(x+y)\circ z = x\circ z + y\circ z$$
$$x\circ y = y\circ x$$

b) Let $x_i > 0$ for all i, $y_j > 0$ for all j, and $\sum_{i=1}^{2s+1} x_i = \sum_{j=1}^{2s+1} y_j = 1$, then $\sum_{i=1}^{2s+1} (x\circ y)_i = 1$.

Proof. $\sum_{i=1}^{2s+1} (x\circ y)_i = \left(\sum_{i=1}^{2s+1} x_i\right)\left(\sum_{j=1}^{2s+1} y_j\right) = 1$.

Example. Using collision algebra S^3, (3) is expressed as

$$(4) \qquad \frac{\partial}{\partial t}\left(\sum_{i=1}^{3} P_i(t)E_i\right) = \left(\sum_{i=1}^{3} P_i(t)E_i\right)\circ\left(\sum_{i=1}^{3} P_i(t)E_i\right) - \left(\sum_{i=1}^{3} P_i(t)E_i\right)$$

where E_i corresponds to species i.

We consider S^{2s+1} and extend (4) for the case of $2s+1$ species as

$$(5) \qquad \frac{\partial}{\partial t}\left(\sum_{i=1}^{2s+1} P_i(t)E_i\right) = \left(\sum_{i=1}^{2s+1} P_i(t)E_i\right)\circ\left(\sum_{i=1}^{2s+1} P_i(t)E_i\right) - \left(\sum_{i=1}^{2s+1} P_i(t)E_i\right).$$

For this equation we see $\frac{\partial}{\partial t} \log \prod_{i=1}^{2s+1} P_i(t) = 0$.

Since,

$$(6) \qquad \frac{\partial}{\partial t} \sum_{i=1}^{2s+1} P_i(t)E_i = \left(\sum_{i=1}^{2s+1} P_i(t)E_i\right)\circ\left(\sum_{i=1}^{2s+1} P_i(t)E_i\right) - \left(\sum_{i=1}^{2s+1} P_i(t)E_i\right)$$
$$= \sum_{i=1}^{2s+1} P_i(t)\left(-\sum_{j=i+1}^{i+s} P_j + \sum_{j=i-s}^{i-1} P_j\right)E_i$$

where $P_l = P_m$ iff $l-m \equiv 0 \pmod{2s+1}$.

$$(7) \qquad \sum_{i=1}^{2s+1} \frac{1}{P_i(t)} \frac{\partial}{\partial t} P_i(t) = \sum_{i=1}^{2s+1} \left(-\sum_{j=i+1}^{i+s} P_j + \sum_{j=i-s}^{i-1} P_j\right) = 0.$$

So we see

$$\frac{\partial}{\partial t} \log \prod_{i=1}^{2s+1} P_i(t) = 0.$$

§ 3. The system based on triple collisions. Using the collision algebra S^{2s+1}, we can naturally consider an equation based on triple collisions. At first we give a theorem for the equation. In the later example, we consider the meaning of the equation in the case of three species.

Theorem. *Put* $P = \sum_{i=1}^{2s+1} P_i(t)E_i$, *and consider*

$$(8) \qquad \frac{\partial}{\partial t} P = P\circ(P\circ P) - P.$$

Then

$$\frac{\partial}{\partial t} \log \prod_{i=1}^{2s+1} P_i(t) \geqq 0,$$

where the equality holds iff $P_i = \frac{1}{2s+1}$ *for all* i.

Proof. (8) is reformed to

$$(9)\quad \frac{\partial}{\partial t}P=P\circ(P\circ P)-P\circ P+P\circ P-P=P\circ(P\circ P-P)+P\circ P-P.$$

In the following line we consider $P_l=P_m$ iff $l-m\equiv 0 \pmod{2s+1}$.
From (6),

$$(10)\qquad P\circ P-P=\sum_{i=1}^{2s+1}P_i\Big(-\sum_{j=i+1}^{i+s}P_j+\sum_{j=i-s}^{i-1}P_j\Big)E_j.$$

So

$$\sum_{i=1}^{2s+1}\frac{(P\circ P-P)_i}{P_i}=0.$$

Considering (6),

$$(11)\qquad P\circ(P\circ P-P)=\Big(\sum_{i=1}^{2s+1}P_iE_i\Big)\circ\Big(\sum_{i=1}^{2s+1}P_i\Big(-\sum_{j=i+1}^{i+s}P_j+\sum_{j=i-s}^{i-1}P_j\Big)E_i\Big)$$
$$=\sum_{i=1}^{2s+1}\alpha_iE_i+\sum_{i=1}^{2s+1}\beta_iE_i$$

where

$$\sum_{i=1}^{2s+1}\alpha_iE_i=\sum_{i=1}^{2s+1}P_i\Big(\sum_{l=i-s}^{i}P_l\Big(-\sum_{j=l+1}^{l+s}P_j+\sum_{j=l-s}^{l-1}P_j\Big)\Big)E_i$$

and

$$\sum_{i=1}^{2s+1}\beta_iE_i=\sum_{i=1}^{2s+1}\Big(\sum_{j=i-s}^{i-1}P_j\Big)P_i\Big(-\sum_{j=i+1}^{i+s}P_j+\sum_{j=i-s}^{i-1}P_j\Big)E_i.$$

$$\sum_{i=1}^{2s+1}\frac{\alpha_i}{P_i}=\sum_{i=1}^{2s+1}\Big(\sum_{l=i-s}^{i}P_l\Big(-\sum_{j=l+1}^{s+l}P_j+\sum_{j=l-s}^{l-1}P_j\Big)\Big)$$
$$=(s+1)\sum_{i=1}^{2s+1}(P\circ P-P)_i.$$

So from property b in section 2, $\sum_{i=1}^{2s+1}\frac{\alpha_i}{P_i}=0$. Consider $\sum_{i=1}^{2s+1}\frac{\beta_i}{P_i}$.

$$(12)\qquad \sum_{i=1}^{2s+1}\frac{\beta_i}{P_i}=\sum_{i=1}^{2s+1}\Big(\Big(\sum_{j=i-s}^{i-1}P_j\Big)\Big(-\sum_{j=i+1}^{i+s}P_j+\sum_{j=i-s}^{i-1}P_j\Big)\Big).$$

Put

$\sum_{j=i-s}^{i-1}P_j=S_i$. Then

$$\sum_{j=i+1}^{i+s}P_j=S_{i+s+1}.$$

So (12) is expressed as

$$\sum_{i=1}^{2s+1}\frac{\beta_i}{P_i}=\sum_{i=1}^{2s+1}S_i(-S_{i+s+1}+S_i)=\frac{1}{2}\sum_{i=1}^{2s+1}(S_i-S_{i+s+1})^2\geqq 0.$$

$$\frac{\partial}{\partial t}\log\prod_{i=1}^{2s+1}P_i(t)$$
$$=\sum_{i=1}^{2s+1}\frac{1}{P_i(t)}\frac{\partial}{\partial t}P_i(t)$$
$$=\sum_{i=1}^{2s+1}\frac{1}{P_i(t)}\{P\circ(P\circ P-P)\}_i+\sum_{i=1}^{2s+1}\frac{1}{P_i(t)}(P\circ P-P)_i$$
$$=\sum_{i=1}^{2s+1}\frac{\beta_i}{P_i(t)}=\frac{1}{2}\sum_{i=1}^{2s+1}(S_i-S_{i+s+1})^2\geqq 0.$$

The equality holds iff $S_i = S_{i+s+1}$ for all i, that is, iff $P_i = \dfrac{1}{2s+1}$ for all i.

Corollary. *Consider for* $0 \leq C_1 \leq 1$

$$\frac{\partial}{\partial t} P = (1-C_1) P \circ P + C_1 P \circ (P \circ P) - P. \tag{13}$$

Then

$$\frac{\partial}{\partial t} \log \prod_{i=1}^{2s+1} P_i(t) \geqq 0.$$

The equality holds iff $P_i = \dfrac{1}{2s+1}$ *for all* i.

Proof. (13) is reformed to

$$\frac{\partial}{\partial t} P = C_1 (P \circ (P \circ P) - P \circ P) + P \circ P - P.$$

So we prove in the same way to the Theorem.

Example. *The case of three species.* We consider the model which satisfies 1) and 4) of section 2, while 2) and 3) are replaced by the following 2)′ and 3)′ respectively.

2)′ Each particle participates in triple collision on the average dt times per time length dt. A triple collision consists of successive three binary collisions as the Fig. 1, in which the particle A collides with the particle B, the B collides with the particle C and the C collides with the particle A. For example, consider the case that A is a particle of species 1, B is a particle of species 2 and C is a particle of species 3, then the A that is a particle of species 1 changes to a particle of species 2 by the first binary collisions, the B that is a particle of species 2 changes to a particle of species 3 by the second binary collisions, and the A that is a particle of species 2 changes to a particle of species 3 by the final binary collisions.

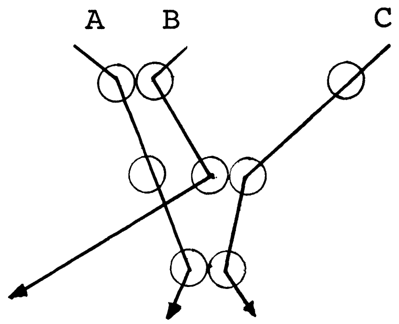

Fig. 1

3)′ Each colliding triple is equally likely chosen.

From the above situation, we have the following equation with

$P(t)=\sum_{i=1}^{3} P_i(t)E_i$ for $P(t) \in S^3$.

$$\frac{\partial}{\partial t}P(t)=(P(t)\circ P(t))\circ P(t)-P(t).$$

Each of ndt particles participates in triple collision in time interval dt, and takes part as the A or the B or the C in Fig. 1. So the above equation is reasonable.

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