78. *On Deformations of the Calabi-Eckmann Manifolds*

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(Comm. by Kunihiko KODAIRA, M. J. A., June 3, 1975)

1. The purpose of this note is to give an explicit construction of an effectively parametrized and complete family of deformations of usual Calabi-Eckmann manifolds.

E. Calabi and B. Eckmann [1] constructed a class of simply connected non-Kähler complex manifolds which are homeomorphic to the product of two odd-dimensional spheres $S^{2n+1}\times S^{2m+1}$. They are the simplest examples of so-called non-Kähler C-manifolds, i.e., simply connected compact complex homogeneous spaces. Deformations of compact complex homogeneous spaces have been studied by several authors. First Kodaira-Spencer [4] constructed an effectively parametrized and complete family of deformations of Hopf surfaces. M. Ise [3] and P. A. Griffiths [2] calculated $H^{\nu}(M,\Theta)$ for C-manifolds M. The latter proved that for each abelian Lie subalgebra of $H^1(M,\Theta)$, there exists a family of deformations of M corresponding to it by the Kodaira-Spencer map. However his method was quite implicit, using similar methods as in Kuranishi's proof of the existence of versal deformations and, in particular, does not give sufficient informations on deformed structures. In the sequel we shall construct an effectively parametrized and complete family of deformations of the Calabi-Eckmann manifolds, using Tits' construction of homogeneous spaces, and state some properties of deformed structures. It seems that our method may be applied to construction of families of deformations of more general non-Kähler C-manifolds. In what follows Θ denotes the sheaf of germs of holomorphic vector fields, and I_n the unit matrix in $GL(n,\boldsymbol{C})$.

2. Tits' construction of the Calabi-Eckmann manifolds. We define a Calabi-Eckmann manifold, following J. Tits [5]. For each $t\in\boldsymbol{C}$, let g_t^λ be the biholomorphic automorphism of $\boldsymbol{C}^p-(0)\times\boldsymbol{C}^q-(0)$ which maps (z,w) to $(z\exp t, w\exp\lambda t)$, where z and w are, resp., the standard coordinates of $\boldsymbol{C}^p$ and $\boldsymbol{C}^q$, and λ is a fixed complex number with $\operatorname{Im}\lambda\neq 0$. Let G_λ be the one-parameter complex Lie group consisting of g_t^λ's. G_λ operates freely and properly on $\boldsymbol{C}^p-(0)\times\boldsymbol{C}^q-(0)$, hence we can construct the quotient manifold $M=\boldsymbol{C}^p-(0)\times\boldsymbol{C}^q-(0)/G_\lambda$. M is called a Calabi-Eckmann manifold. The natural projection from $\boldsymbol{C}^p-(0)\times\boldsymbol{C}^q-(0)$ to $\boldsymbol{P}^{p-1}\times\boldsymbol{P}^{q-1}$ makes M a complex analytic fibre bundle over $P^{p-1}\times P^{q-1}$ whose fibre is an elliptic curve. Note also that, by the projection to each factor of $P^{p-1}\times P^{q-1}$, M is a fibre bundle over a projective space whose fibre is a homogeneous Hopf manifold.

3. Construction of a family of deformations of M. Let M and λ be as in 2. We shall construct a family of deformations of M. Let (A, B) be an element of $GL(p, C)\times GL(q, C)$, $g_t^{(A,B)}$ an automorphism of $C^p-(0)\times C^q-(0)$ defined by

$$g_t^{(A,B)}(z, w)=(z \exp(At), w \exp(Bt)),$$

and $G_{(A,B)}$ the one-parameter group $\{g_t^{(A,B)}\}_{t\in C}$. $g_t^{(I_p,\lambda I_q)}$ is nothing but g_t^λ in 2. Then we get

Proposition 1. *Assume that (A, B) belongs to a sufficiently small neighbourhood $\tilde{U}$ of $(I_p, \lambda I_q)$ in $GL(p, C)\times GL(q, C)$. Then $G_{(A,B)}$ operates freely and properly on $C^p-(0)\times C^q-(0)$, and the quotient space $M_{(A,B)}=C^p-(0)\times C^q-(0)/G_{(A,B)}$ is a compact complex manifold which is homeomorphic to M. Moreover $\tilde{\mathcal{M}}=\{M_{(A,B)}\}_{(A,B)\in\tilde{U}}$ forms a complex analytic family of deformations of M over $\tilde{U}$.*

In fact the properness of the action of $G_{(A,B)}$ on $C^p-(0)\times C^q-(0)$ is proved by a simple calculation using Jordan normal forms of A and B. $\tilde{\mathcal{M}}$ is defined as follows: Let g_t be an automorphism of $C^p-(0)\times C^q-(0)\times\tilde{U}$ defined by

$$g_t(z, w, (A, B))=(g_t^{(A,B)}(z, w), (A, B)),$$

and $G=\{g_t\}_{t\in C}$. Then $\tilde{\mathcal{M}}=C^p-(0)\times C^q-(0)\times\tilde{U}/G$ with the natural projection π from $\tilde{\mathcal{M}}$ to $\tilde{U}$ is a complex analytic family of deformations of $M=M_{(I_p,\lambda I_q)}$ over $\tilde{U}$. Note that, for any $\mu\in C^*$, $M_{(A,B)}$ and $M_{(\mu A,\mu B)}$ are isomorphic to each other. Let U be a submanifold of $\tilde{U}$ defined by $\det A=1$. We shall prove that the subfamily $\mathcal{M}=\pi^{-1}(U)\rightarrow U$ gives an effectively parametrized and complete family of deformations of M.

4. Some lemmata. Let M be as in 2.

Lemma 1. *We have*

$$\dim H^1(M, \Theta)=\dim H^0(M, \Theta)=p^2+q^2-1,$$
$$\dim H^\nu(M, \Theta)=0 \qquad \text{for } \nu\geqq 2.$$

Moreover there exists a natural isomorphism

$$H^1(M, \Theta)\simeq H^0(M, \Theta)\otimes H^1(M, \mathcal{O})$$

and Aut (M) *operates trivially on* $H^1(M, \mathcal{O})$.

For a proof see M. Ise [3].

Let $V=C^n-(0)/\{\alpha^k I_n\}_{k\in Z}$ be an n-dimensional Hopf manifold, where $0<|\alpha|<1$, $\mathcal{U}$ a sufficiently small neighbourhood of αI_n in $GL(n, C)$, and $\mathcal{V}=C^n-(0)\times\mathcal{U}/\{h^n\}_{n\in Z}$, where h is an automorphism of $C^n-(0)\times\mathcal{U}$ defined by

$$h(z, u)=(uz, u),$$

where $u\in\mathcal{U}$, z is the natural coordinate in C^n, and u operates linearly on C^n. $\mathcal{V}$ is a complex analytic family of deformations of V over $\mathcal{U}$.

Lemma 2. *The family $\mathcal{V}$ is complete and effectively parametrized at $u=\alpha I_n$.*

Proof. See Kodaira-Spencer [4], where the lemma is proved for $n=2$. The generalization for $n\geqq 3$ is straightforward.

5. Proof of Theorem and some corollaries.

Theorem. *The family $\mathcal{M}\to U$ defined in 3 gives an effectively parametrized and complete family of deformations of M.*

Proof. Since $\dim U=\dim H^1(M,\Theta)=p^2+q^2-1$, we have only to show that the Kodaira-Spencer map ρ from $T_0(U)$ to $H^1(M,\Theta)$ is surjective, where $T_0(U)$ is the tangent space of U at $(I_p,\lambda I_q)$. Consider the subfamily $\mathcal{M}'=\{M_{(A,B)}\mid A=I_p\}$. Let $U'=\{(A,B)\in U\mid A=I_p\}$. U' is a submanifold of U. We shall show that ρ maps $T_0(U')$ injectively into $H^1(M,\Theta)$. Since $A=I_p$ for $M_{(A,B)}\in\mathcal{M}'$, there exists a holomorphic map φ from $M_{(A,B)}$ to $\boldsymbol{P}^{p-1}$ defined in an obvious manner. φ makes $M_{(A,B)}\in\mathcal{M}'$ a holomorphic fibre bundle over $\boldsymbol{P}^{p-1}$ whose fibre is a Hopf manifold. Hence the family $\mathcal{M}'$ over U' preserves the fibre structure of M over $\boldsymbol{P}^{p-1}$. We have the following exact sequence on M associated with φ:

$$0\longrightarrow\Theta_{M/P^{p-1}}\longrightarrow\Theta_M\longrightarrow\varphi^*\Theta_{P^{p-1}}\longrightarrow 0,$$

where $\Theta_{M/P^{p-1}}$ is the sheaf of holomorphic vector fields tangential to the fibres of φ. From this follows the existence of an exact sequence of cohomology groups of M,

$$\cdots\longrightarrow H^0(\Theta_M)\xrightarrow{j}H^0(\varphi^*\Theta_{P^{p-1}})\longrightarrow H^1(\Theta_{M/P^{p-1}})$$
$$\xrightarrow{i}H^1(\Theta_M)\longrightarrow H^1(\varphi^*\Theta_{P^{p-1}})\longrightarrow H^2(\Theta_{M/P^{p-1}})\longrightarrow 0.$$

Lemma 3. *The map $j\colon H^0(\Theta_M)\longrightarrow H^0(\varphi^*\Theta_{P^{p-1}})$ is surjective, and $H^2(\Theta_{M/P^{p-1}})=0$.*

In fact, since M is homogeneous, j is surjective. By Leray's spectral sequence, we see easily that $H^2(\Theta_{M/P^{p-1}})=0$.

Since $\mathcal{M}'$ preserves the fibre structure by φ, $\rho|_{T_0(\mathcal{U}')}$ maps $T_0(\mathcal{U}')$ into $H^1(\Theta_{M/P^{p-1}})$. By the above lemma, i is injective.

Lemma 4. *There exists a natural isomorphism*

$$H^1(M,\Theta_{M/P^{p-1}})\simeq H^1(F,\Theta_F),$$

where F is a fibre of φ which is a homogeneous Hopf manifold.

In fact, by Leray's spectral sequence, we have an exact sequence

$$0\longrightarrow H^1(\boldsymbol{P}^{p-1},\varphi_*\Theta_{M/P^{p-1}})\longrightarrow H^1(M,\Theta_{M/P^{p-1}})$$
$$\xrightarrow{\beta}H^0(\boldsymbol{P}^{p-1},R^1\varphi_*\Theta_{M/P^{p-1}})\longrightarrow H^2(\boldsymbol{P}^{p-1},\varphi_*\Theta_{M/P^{p-1}}).$$

By a simple calculation $R^1\varphi_*\Theta_{M/P^{p-1}}\simeq\mathcal{O}_{P^{p-1}}\otimes_C V$, where $V\simeq H^1(F,\Theta_F)$ and $\varphi_*\Theta_{M/P^{p-1}}\simeq\mathcal{O}^{q^2}$. Hence $E_2^{1,0}=E_2^{2,0}=0$, and $H^1(M,\Theta_{M/P^{p-1}})\simeq H^0(\boldsymbol{P}^{p-1},R^1\varphi_*\Theta_{M/P^{p-1}})\simeq H^1(F,\Theta_F)$. Moreover β is given by the restriction to a fibre of φ. q.e.d.

By the construction, we have a natural projection ϖ from $\mathcal{M}'$ to $\boldsymbol{P}^{p-1}\times U'$. Then, for any point $w\in\boldsymbol{P}^{p-1}$, $\varpi^{-1}(\{w\}\times U')$ is a family of deformations of Hopf manifolds over $\{w\}\times U'$, which is effectively parametrized and complete at $\{w\}\times(I_p, \lambda I_q)$. $\beta\circ\varphi|_{T_0(U')}$ maps $T_0(U')$ into $H^1(F_w, \Theta_{F_x})$ where $F_w=\varphi^{-1}(w)$. Hence, by the completeness of $\varpi^{-1}(\{w\}\times U')$, $\beta\circ\rho$ is surjective. Since β is an isomorphism, $\rho|_{T_0(U')}$ maps $T_0(U')$ isomorphically onto $H^1(M, \Theta_{M/P^{p-1}})$. (In the above argument we identify $H^1(M, \Theta_{M/P^{p-1}})$ with its isomorphic image in $H^1(M, \Theta)$.) Quite similarly we have a subfamily $\mathcal{M}''$ of $\mathcal{M}$ over U'', preserving the fibre structure over P^{q-1}, where $U''=\{(A,B)\in U\,|\,B$ is a scalar matrix$\}$, and the bijective map $\rho|_{T_0(U'')}$ from $T_0(U'')$ to $H^1(M, \Theta_{M/P^{q-1}})$ which is identified with the subspace of $H^1(M, \Theta)$. By Lemma 1 in 3, these two subspaces are the only invariant subspaces by the operation of Aut (M) of dimension greater than 1, and generator the vector space $H^1(M, \Theta)$. Hence, by the linearity of ρ, we see that ρ maps $T_0(U)$ surjectively to $H^1(M, \Theta)$,

q.e.d.

We shall state some results easily deduced from the above theorem.

Proposition 2. *For $(A,B)\in U'$ sufficiently near to $(I_p, \lambda I_q)$, we have the following equalities:*

(1) $\dim H^0(M_{(A,B)}, \Theta)=\dim H^1(M_{(A,B)}, \Theta)$

$$=\dim_C \{(P,Q)\in M_p(C)\times M_q(C)\,|\,PA=AP, QB=BQ\}-1,$$

where $M_n(C)$ is the vector space of complex $(n\times n)$-matrices.

(2) *$M_{(A,B)}$ is biholomorphic to $M_{(A',B')}$ if and only if there exist non-singular matrices $P\in GL(p,C)$ and $Q\in GL(q,C)$ such that $P^{-1}AP=A'$ and $Q^{-1}BQ=B'$.*

(3) *Let $\mathrm{a}(M)$ be the transcendence degree of the field of meromorphic functions on M, and n_A the dimension of the vector space over $\boldsymbol{Q}$ generated by the eigenvalues of A. Then we have*

$$\mathrm{a}(M_{(A,B)})=p+q-n_A-n_B.$$

Remark 1. $M_{(A,B)}$ is a non-Kähler prehomogeneous manifold.

Remark 2. There exists a (non-small) deformation of M which is not isomorphic to any $M_{(A,B)}$ above.

References

[1] E. Calabi and B. Eckmann: A class of compact complex manifolds which are not algebraic. Ann. of Math., **58**, 494–500 (1953).

[2] P. Griffiths: Some geometric and analytic properties of homogeneous complex manifolds. Acta Math., **110**, 115–208 (1963).

[3] M. Ise: Some properties of complex analytic vector bundles over compact complex homogeneous spaces. Osaka Math. J., **12**, 217–252 (1960).

[4] K. Kodaira and D. C. Spencer: On deformations of complex analytic structures. II. Ann. of Math., **67**, 403–466 (1958).

[5] J. Tits: Espaces homogènes complexes compacts. Comment. Math. Helv., **37**, 111–120 (1962–1963).