113. Normalized Series of Prestratified Spaces

Complex Analytic De Rham Cohomology. IV

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In this note we introduce,[1] for analytic varieties, a type of series of prestratified spaces, which we call a *normalized series of prestratified spaces* (or simply a *normalized series*, when there is no fear of confusions). We also state an existence theorem on such a series. We stated two basic quantitative properties of analytic varieties in $[4]_2$. It is this notion of normalized series that constitutes basis of the discussions for the results in $[4]_2$.

Basic ideas. Let V be an algebraic or analytic variety.[2] The basic theorems: Weierstrass's preparation theorem and Noether's normalization theorem represent the variety V as a (finite) *ramified covering* of an another variety V', which has simpler properties than V. In both theorems the study of *the ramification locus* W of the covering map $\pi: V \to V'$ has important meanings for the study of the variety V. Of course, $\dim W < \dim V$, and we may say that the above theorems enable us *inductive discussions* of varieties on the dimension of varieties in question. We note, moreover, that the above theorems attach to the given variety V a set of functions, which is basic in the study of the variety V.

Now our hope in introducing the notion of normalized series is to systematize ideas[3] in the above theorems (and methods of ramified maps in general): Let V be an analytic variety. Then a *normalized series attached to* V consists of series $\mathfrak{R}$ of varieties, prestratified spaces, $\cdots$ and $\mathfrak{F}$ of collections of analytic functions (cf. n. 2). Varieties and strata appearing in the series $\mathfrak{R}$ are basically related to each other by ramified maps (arising naturally from the series $\mathfrak{R}$).

By attaching to the given variety V a *series* of varieties, prestratifications, $\cdots$ instead of a single variety (as in standard treatments of basic theorems mentioned above), we can discuss, systematically, the variety V inductively on the dimension of varieties, $\cdots$ (appearing

1) We use the same notions and notations as in $[4]_1$, $[4]_2$ and $[4]_3$. In particular we use the notion of prestratified spaces in the sense in $[4]_3$.

2) Except the part explaining basic ideas in the introduction, analytic varieties and analytic functions are always *real* analytic ones.

3) Ideas understood as explained just before.

in the series $\mathfrak{R}$). Among conditions imposed on the normalized series $(\mathfrak{R}, \mathfrak{F})$, *the higher discriminant condition* $(9)_3$ analyzes, in details, singular loci of ramified maps in question. This condition plays important roles in the discussions of quantitative properties of analytic varieties (cf. $[4]_2$).

The notion of normalized series is originally defined to prove the results stated in $[4]_2$. However, we point out that the notion of normalized series concerns basic properties of analytic varieties, which may be meaningful in wider situations than in $[4]_2$.

n.1. Auxiliary notions. We will introduce certain auxiliary notions used in discussions of distance properties of analytic varieties: Let $\boldsymbol{R}^n$ be a euclidean space and U a bounded domain in $\boldsymbol{R}^n$. For a positive number r, let $N_r(U)$ denote the neighborhood of U as follows: $N_r(U)=\bigcup_P \Delta(r; P)$,[4] where $P \in U$. Let U, U' be bounded domains in $\boldsymbol{R}^n$. We say that U' is a d-envelop of U if

(1) $N_{r_1}(N_r(U)) \subset U'$, where r, r_1 are the radius[5] of $U, N_r(U)$.

We mean by a *triplet in* $\boldsymbol{R}^n$ a collection $Q=(U, V, \mathcal{S}_0)$ consisting of a bounded domain U in $\boldsymbol{R}^n$, a variety V in U and a prestratification $\mathcal{S}_0$ of (U, V).[6]

Let $Q=(U, V, \mathcal{S}_0)$, $Q'=(U', V', \mathcal{S}_0')$ be triplets in $\boldsymbol{R}^n$. We say that Q' is a *d-envelop of* Q if the following are valid.

$(2)_1$ U' is a d-envelop of U and $V=V' \cap U$.

$(2)_2$ $\mathcal{S}_0$ is the restriction of $\mathcal{S}_0'$ to U. Moreover, the restriction map $Rs: \mathcal{S}_0' \ni S' \to \mathcal{S}_0 \ni S=S' \cap U$ is bijective.

We say, moreover, that (Q, Q') satisfies *d-separation condition* if Q' is a d-envelop of Q and (Q, Q') satisfies the following:

$(3)_1$ For any $(S_1, S_2) \in \mathcal{S}_0 \times \mathcal{S}_0$ such that $S_1 \not< S_2$,

$N_\delta(S_1, \text{from } S_1') \cap S_2' = \phi$ with a suitable δ.

$(3)_2$ For any $S \in \mathcal{S}_0$, $\{N_\delta(S, \text{from } S') \cap U\}_\delta \sim \{N_{\delta'}(S' \cap N_r(U), \text{from } S')\}_{\delta'}$.

In the above $S', S_1', \cdots$ denotes $Rs^{-1}(S)$, $Rs^{-1}(S_1)$, $\cdots$ Moreover, we denote by r the radius of U.

n.2. Admissible series of prestratified spaces. Let $\boldsymbol{R}^n(x)$ be a euclidean space with a system $(x)=(x_1, \cdots, x_n)$ of coordinates. We introduce the following

Definition 1. An *admissible series* $\mathfrak{R}$ *in* $\boldsymbol{R}^n(x)$ is a collection as follows:

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4) See $[4]_1$.

5) $r=\sup_{P,P'} d(P, P')$, where $P, P' \in U, \cdots$

6) $\mathcal{S}_0$ is a prestratification of U such that V is the union of strata of $\mathcal{S}_0$. Let $\mathcal{S}$ denote the collection: $\{S \in \mathcal{S}_0; S \subset V\}$. We call $\mathcal{S}$ *the prestratification of* V *induced from* $\mathcal{S}_0$.

7) This equivalence means the following: Given a couple $\delta(\delta')$, there exists a couple $\delta'(\delta)$ so that $N_\delta(S, \text{from } S') \cap U \supset N_{\delta'}(S' \cap N_r(U), \text{from } S') \cap U$ $(N_\delta(S' \cap N_r(U), \text{from } S') \cap U \subset N_{\delta'}(S, \text{from } S') \cap U.)$

$(4)_1$ A system $(y)=(y_1, \cdots, y_n)$ of coordinates of $\boldsymbol{R}^n$.

$(4)_2$ Series $Q=\{Q^j\}_{j=1}^n$, $Q'=\{Q'^j\}_{j=1}^n$ of triplets $Q^j=(U^j, V^j, \mathcal{S}_0^j)$, $Q'^j=(U'^j, V'^j, \mathcal{S}_0'^j)$ in $\boldsymbol{R}^j(y^j)$[8].

The data $(y), Q, Q'$ are required to satisfy the following:

$(5)_1$ Q'^j is a d-envelop of Q^j, and (Q^j, Q'^j) satisfies d-separation condition $(j=1, \cdots, n)$.

$(5)_2$ U^j, U'^j are connected and any $S \in \mathcal{S}_0$ $(S' \in \mathcal{S}_0'^j)$ is a connected analytic manifold $(j=1, \cdots, n)$.

$(5)_3$ Each stratum $S \in \mathcal{S}_0 - \mathcal{S}(S' \in \mathcal{S}_0' - \mathcal{S}')$ is a connected component of $U^j - V^j$ $(U'^j - V'^j)$ and vice versa, $j=1, \cdots, n$. Here $\mathcal{S}^j(\mathcal{S}'^j)$ denotes the prestratification of $V(V')$ induced from $\mathcal{S}_0^j(\mathcal{S}_0'^j)$.

$(6)_1$ $U^j(U'^j) \cong U^{j-1} \times I(U'^{j-1} \times I')$, $j=2, \cdots, n$, where I, I' are open segments such that $I \subsetneqq I'$.

$(6)_2$ For any $S^j \in \mathcal{S}^j(S'^j \in \mathcal{S}'^j)$, $\pi_{j-1j}(S^j)(\pi_{j-1j}(S'^j))$ is a stratum of $\mathcal{S}_0^{j-1}(\mathcal{S}_0'^{j-1})$, $j=2, \cdots, n$. Moreover, $\pi_{j-1j}: S^j \to \pi_{j-1j}(S^j)$[9] $(\pi_{j-1j}: S'^j \to \pi_{j-1j}(S'^j))$ is *real analytically biholomorphic.*[10]

Remark. Among conditions in (5), (6), the condition $(6)_2$ is noteworthy. The *biholomorphic assertion* of the restriction of π_{j-1j} to strata of $\mathcal{S}^j(\mathcal{S}'^j)$ plays important roles in inductive discussion of the triplet Q^j, Q'^j on $j=1, \cdots, n$.

n.3. Normalized series of prestratified spaces. Let $\boldsymbol{R}^n(x)$ be a euclidean space with coordinates (x), and let $\mathfrak{R}=((y), Q, Q')$ be an admissible series in $\boldsymbol{R}^n(x)$, where $Q=\{Q^j\}_{j=1}^n$, $Q'=\{Q'^j\}_{j=1}^n$ are explicitly as follows: $Q^j=(U^j, V^j, \mathcal{S}_0^j)$, $Q'^j=(U'^j, V'^j, \mathcal{S}_0'^j)$, $j=1, \cdots, n$. Let $S'^j \in \mathcal{S}'^j$, where $\mathcal{S}'^j$ is the prestratification of V'^j induced from $\mathcal{S}_0'^j$. We denote the dimension of S'^j by $\tilde{n}$. We introduce the following

Definition 2. A *representation datum* $\mathfrak{f}(S'^j)$ of S'^j is a pair $\{f(S'^j), f'(S'^j)\}$ as follows:

$(7)_1$ A set $f(S'^j)=\{f_t(S'^j)\}_{t=1}^{j-\tilde{n}}$, where $f_t(S'^j)$ is a monic polynomial in $y_{\tilde{n}+t}$ with coefficients $f_{tu}(y_1, \cdots, y_{\tilde{n}})$'s. Here f_{tu}'s are analytic functions in $U'^{\tilde{n}}$.

$(7)_2$ A set $f'(S'^j)=\{f'_s(S'^j)\}_{s=1}^{\tilde{s}}$, where $\tilde{s} \geqq j-\tilde{n}$ and $f'_s(S'^j)$'s are analytic functions in U'^j.

The sets $f(S'^j), f'(S'^j)$ must vanish on S'^j.

Varieties attached to representation datum. (i) We denote by $V(f(S'^j)), V(f'(S'^j))$ the zero loci of $f(S'^j), f'(S'^j)$ in U'^j.

(ii) *The ramification locus of* $\mathfrak{f}'(S'^j)$: We define the ramification

8) $\boldsymbol{R}^j(y^j)$ denotes the linear subspace: $y_{j+1}=\cdots y_n=0$ $(j=1, \cdots, n)$.

9) π_{j-1j} denotes the natural projection from $\boldsymbol{R}^j(y^j)$ to $\boldsymbol{R}^{j-1}(y^{j-1})$.

10) In the complex analytic case, the notion of normalized series can be defined in a paralell manner to the real analytic case. However, the essential difference in the complex analytic case is that one should replace the *biholomorphic property* of π_{j-1j}'s $(6)_2$ by *locally biholomorphic properties* of π_{j-1j}'s. The condition $(6)_2$ seems to be a peculiar advantage in the real analytic case.

locus $\Delta(f'(S'^j))$ of $f'(S'^j)$ to be the zero locus on $V(f'(S'^j))$ of the following functions:

(a) $\left\{\left|\dfrac{\partial f'^I(S'^j)}{\partial(y_{\tilde{n}+1},\cdots,y_j)}\right|\right\}_I$, where $f'^I(S'^j)=(f_{i_1},\cdots,f_{i_{j-\tilde{n}}})$ with $I=(i_1,\cdots,i_{j-n})$ and $f_{i_1},\cdots\in f'(S'^j)$.

(iii) *Higher discriminant loci of* $f(S'^j)$: Let $m=(m_1,\cdots,m_{j-\tilde{n}})\in Z^{+j-\tilde{n}}$. We define *the m-th discriminant locus* $D_m(f(S'^j))$ of S'^j to be the *locally closed* analytic variety in U'^j as follows:

(b) $D_m(f(S'^j))=\{Q'^j\in \boldsymbol{R}^j(y^j);\ D_{\tilde{m}_t}f_t(Q'^j)=0,\quad 0\leqq \tilde{m}_t<m_t-1,\ D_{m_t}f_t(Q'^j)\neq 0\ (t=1,\cdots,j-n)\}$.

In (b) we denote by D_{m_t} the differential operator: $\partial^{m_t}/\partial y^{m_t}_{\tilde{n}+t}$.

We call a collection $\{\mathfrak{f}(S'^j);S'^j\in\mathcal{S}'^j\}$ *a representation datum of* (Q^j,Q'^j), where $\mathfrak{f}(S'^j)$ is a representation datum of S'^j. Moreover, we call a series $\mathfrak{F}=\{\mathfrak{F}^j\}_{j=1}^n$ *a representation datum of* $\mathfrak{R}$ if $\mathfrak{F}^j$ is a representation datum of (Q^j,Q'^j), $j=1,\cdots,n$.

Now let $\mathfrak{R}=((y,Q,Q')$ be an admissible series in $\boldsymbol{R}^n(x)$, where $Q=\{Q^j\}_{j=1}^n$, $Q'^j=\{Q'^j\}_{j=1}^n$ are explicitly as follows:

(8) $Q^j=(U^j,V^j,\mathcal{S}_0^j)$, $Q'^j=(U'^j,V'^j,\mathcal{S}_0'^j)$.

Moreover, let $\mathfrak{F}=\{\mathfrak{F}^j\}_{j=1}^n$ be a representation datum of $\mathfrak{R}$, where $\mathfrak{F}^j$ is explicitly as follows:

(8)′ $\mathfrak{F}^j=\{(f(S'^j),f'(S'^j));S'^j\in\mathcal{S}'^j\}$, where $\mathcal{S}'^j$ is the induced prestratification of V'^j (from $\mathcal{S}_0'^j$).

Being $(\mathfrak{R},\mathfrak{F})$ be as above, we introduce the following

Definition 3. The pair $(\mathfrak{R},\mathfrak{F})$ is called a *normalized series of prestratified spaces in* $\boldsymbol{R}^n(x)$ if the following conditions are valid:

$(9)_1$ For any $S'^j\in\mathcal{S}'^j$, $V(f'(S'^j))$ is the union of strata of $\mathcal{S}'^j$ and $\dim V(f'(S'^j))=\dim S'^j (j=1,\cdots,n)$.

$(9)_2$ For any $S'^j\in\mathcal{S}'^j$, $\Delta(f'(S'^j))\cap S'^j=\phi$, $j=1,\cdots,n$.

$(9)_3$ For any pair $(S_1'^j,S_2'^j)\in\mathcal{S}'^j\times\mathcal{S}'^j$ such that $S_1'^j\prec S_2'^j$, there exists a unique element $m\in Z^{+j-\tilde{n}}$ such that

$$S_1'^j\subset D_m(f(S_2'^j)),\qquad j=1,\cdots,n.$$

In $(9)_3$ we denote $\dim S_2'^j$ by $\tilde{n}$.

We call $(9)_2$, $(9)_3$ respectively *ramification* and *higher discriminant conditions*. These conditions play basic roles in our investigations in $[4]_2$. (Cf. the introduction.)

n.4. Normalized series attached to germs of varieties. Let $\boldsymbol{R}^n(x)$ be a euclidean space, and let $P^n\in\boldsymbol{R}^n(x)$. Moreover, let V be a germ of an analytic variety at P^n such that $1\leqq\dim V\leqq n-1$. Furthermore, let $(\mathfrak{R},\mathfrak{F})$ be a normalized series in $\boldsymbol{R}^n(x)$. We write $\mathfrak{R}=((y),Q,Q')$, $\mathfrak{F}=\{\mathfrak{F}^j\}_{j=1}^n$ in the form (8), (8)′. We then introduce the following

Definition 4. The normalized series $(\mathfrak{R},\mathfrak{F})$ is said to be *attached properly to* V if the following are valid:

(10.1) For any $S'^j \in \mathcal{S}_0'^j$, $\bar{S}'^j \ni P^j(=\pi_{jn}(P^n))$, $j=1, \cdots, n$.

(10.2) The germ V coincides with the germ of V^n at P^n.

(10.3) For each irreducible component V_τ of V, there exists a variety V'_τ in U'^n so that (*i*) V'_τ is the union of strata of $\mathcal{S}'^j$ and (ii) the germ of V'_τ at P^n coincides with V_τ.

(10.4) For each $S' \in \mathcal{S}'^j$, $f(S'^j)$ consists of Weierstrass polynomials, $j=1, \cdots, n$.

(10.5) For each $S'^j \in \mathcal{S}'^j$, the germ $V(f'(S'^j))$ of the zero locus of $f'(S'^j)$ at P^j is irreducible. Moreover, the ideal of $V(f'(S'^j))$ is the gemm of $f(S'^j)$ at P^j, $j=1, \cdots, n$.

We will state an existence theorem of normalized series in the following form:

Theorem. *Let $P^n \in \boldsymbol{R}^n(x)$, and let V be a germ of a variety at P^n such that $1 \leqq \dim V \leqq n-1$. Then there exists a normalized series $(\mathfrak{N}, \mathfrak{F})$ attached properly to V.*

Remark. Details of the results in $[4]_1 \sim [4]_3$ and in this note will appear elsewhere. (The author plans to publish first details for the results in $[4]_3$ and in this note in a quite near future.) Earlier versions on the contents in this note will be found in [5], where certain properties of normalized series and the construction of the series $(\mathfrak{N}, \mathfrak{F})$ in Theorem are found. In [5] sharper forms of Theorem will be also found.

References

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11) We refer 'Complex analytic de Rham cohomology I, II and III' to, respectively, as $[4]_1$, $[4]_2$ and $[4]_3$.