112. Serial Endomorphism Rings

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1. Recently, Ringel and Tachikawa [2] have proved that the endomorphism ring of a minimal generator cogenerator module over a serial ring is again serial. In this connection, the purpose of this note is to obtain a necessary and sufficient condition that the endomorphism rings of modules over a serial ring are serial.

Let R be a ring. An R-module M is said to be *serial* if its submodules form a finite chain. We call a ring R *left* (*right*) *serial* if ${}_R R$ (R_R) is a direct sum of serial modules. A left and right serial ring is called *serial*, and this is the same with a generalized uni-serial ring in the sense of Nakayama [1].

A subquotient U of an R-module M will be called *proper* if $U=A/B$ with $M\supsetneqq A\supseteq B\neq 0$, and we shall say that an R-module P appears as a proper subquotient of M if P is isomorphic to a proper subquotient of M. Subquotients U and V of a serial R-module M will be called *joined* if a non-zero submodule of one of U and V coincides with a non-zero factor module of the other of U and V. Let $M_1, \cdots, M_n$ be R-modules. An *iso*-subquotient of M_i will be a proper subquotient of M_i which is isomorphic to some M_j. A *pair*-subquotient of M_i will be a factor module of M_i which is isomorphic to a submodule of some M_j or a submodule of M_i which is isomorphic to a factor module of some M_j. With these definitions we can state the following main results.

Theorem 1. *Let R be a serial ring and $M_1, \cdots, M_n$ indecomposable left R-modules. The following statements are equivalent.*

a) *The endomorphism ring S of $M=M_1\oplus\cdots\oplus M_n$ is serial.*

b) *For each M_i, no iso-subquotient of M_i is joined with any pair-subquotient of M_i.*

As a special case, if there are no iso-subquotients, then the condition b) of Theorem 1 is satisfied, so we have

Corollary 1. *Let R be a serial ring and $M_1, \cdots, M_n$ indecomposable left R-modules. If no M_i appears as a proper subquotient of any M_j, then the endomorphism ring S of $M=M_1\oplus\cdots\oplus M_n$ is serial.*

Since each indecomposable module over a serial ring R is serial, no indecomposable injective or projective R-modules appear as a proper subquotient of any indecomposable R-modules. So, the above corollary can be regarded as a generalization of [2, Lemma 5.6].

Applying Corollary 1, we shall prove the next theorem later on.

Theorem 2. *Let R be a* QF-3 *ring whose minimal faithful left module is a direct sum of serial modules. Then the maximal quotient ring Q of R is serial.*

Here, a QF-3 ring means a ring which has a unique minimal faithful left and right module, respectively, and we do not assume any chain conditions on R.

2. If R is a ring with the radical J and M is a left (right) R-module, we write $T(M)=M/JM$ $(T(M)=M/MJ)$ and $S(M)$ for the socle of M. For an R-module M having the finite composition series, $|M|$ denotes the composition length of M. We shall write homomorphisms on the opposite side to scalars.

Proof of Theorem 1. b)$\Rightarrow$a) We may assume that $M_1, \cdots, M_n$ are mutually non-isomorphic. For a primitive idempotent e of R, let $\mathfrak{F}^e$ denote a set of pairs $\{[A, M_i] \mid 1 \leq i \leq n,\ 0 \neq A \subseteq M_i,\ T(A) \simeq T(Re)\}$. We introduce an order into $\mathfrak{F}^e$ by defining $[A, M_i] \preccurlyeq [B, M_j]$ if there exists a homomorphism $\alpha: M_j \to M_i$ such that $(B)\alpha = A$. We want to show that $\mathfrak{F}^e$ is a disjoint union of linearly ordered components. For this aim, first consider the case $[A, M_i] \succcurlyeq [B, M_j]$ and $[C, M_k] \succcurlyeq [B, M_j]$. Let $\alpha: M_i \to M_j$ and $\beta: M_k \to M_j$ be such that $(A)\alpha = B$ and $(C)\beta = B$. Suppose that no order relation exists between $[A, M_i]$ and $[C, M_k]$. Then, without loss of generality, it may be assumed $|M_i/A| > |M_k/C|$ and $|A| > |C|$. Hence, we can take submodules P and Q of M_i such that $M_i \supsetneqq P \supseteq A \supseteq Q \neq 0$, $|P/A| = |M_k/C|$ and $|A/Q| = |C|$. Then P/Q is a proper subquotient of M_i, and $P/Q \simeq M_k$. Namely, P/Q is an iso-subquotient of M_i, and obviously $M_i/\mathrm{Ker}\,(\alpha)$ $(\subsetneqq M_j)$ is a pair-subquotient of M_i. Since $(A)\alpha = (C)\beta = B \neq 0$, we have $0 \neq |A/\mathrm{Ker}\,(\alpha)| = |B| = |C/\mathrm{Ker}\,(\beta)| \leq |C| = |A/Q|$, hence $M_i \supseteq P \supseteq A \supsetneqq \mathrm{Ker}\,(\alpha) \supseteq Q$. This implies that P/Q and $M_i/\mathrm{Ker}\,(\alpha)$ are joined. So, we have a contradiction. Next, let $[A, M_i] \preccurlyeq [B, M_j]$, $[C, M_k] \preccurlyeq [B, M_j]$ and $\gamma: M_j \to M_i$, $\delta: M_j \to M_k$ be such that $(B)\gamma = A$, $(B)\delta = C$. If we suppose that no order relation exists between $[A, M_i]$ and $[C, M_k]$, then as above we can take submodules P and Q of M_i with $M_i \supsetneqq P \supseteq A \supsetneqq Q \neq 0$, $|P/A| = |M_k/C|$ and $|A/Q| = |C|$. Since $|\mathrm{Im}\,(\gamma)/A| = |M_j/B| = |\mathrm{Im}\,(\delta)/C| \leq |M_k/C| = |P/A|$ and $\mathrm{Im}\,(\gamma) \supseteq A \supsetneqq Q$, we obtain $P \supseteq \mathrm{Im}\,(\gamma) \supsetneqq Q$. This contradicts the assumption that an iso-subquotient P/Q is not joined with a pair-subquotient $\mathrm{Im}\,(\gamma)$. For the remaining cases, the transitive law of orders assures our assertion.

Let $e_i \in S = \mathrm{End}\left(\bigoplus_{i=1}^{n} M_i\right)$ be the projection onto M_i and N the radical of S. We have only to prove that both $e_i N_S$ and ${}_S N e_i$ are quasi-primitive i.e. homomorphic images of primitive ideals. Let $T(M_i) \simeq T(Re)$, that is, $[M_i, M_i] \in \mathfrak{F}^e$. We can choose $[A, M_k] \in \mathfrak{F}^e$ which is small next to $[M_i, M_i]$ in the linearly ordered component of $\mathfrak{F}^e$. Then there exists $p: M_i \to M_k$ with $(M_i)p = A$ and $p \in e_iNe_k$, since $[M_i, M_i] \succ [A, M_k]$. Hence, we have $pS = pe_kS \subseteq e_iN$. To show the inverse inclusion, let $q \in e_iN$ be a given element and $1 \leq j \leq n$ a given index. If $qe_j: M_i \to M_j$ is non-zero, $[M_i, M_i] \succcurlyeq [\mathrm{Im}\,(qe_j), M_j]$. But, since qe_j is non-isomorphism, we obtain $[M_i, M_i] \succ [\mathrm{Im}\,(qe_j), M_j]$. It follows $[A, M_k] \succcurlyeq [\mathrm{Im}\,(qe_j), M_j]$ by the choice of $[A, M_k]$. Thus, there exists $r: M_k \to M_j$ such that $(A)r = \mathrm{Im}\,(qe_j)$. Now, we note $\mathrm{Im}\,(pr) = \mathrm{Im}\,(qe_j)$, hence $\mathrm{Ker}\,(pr) = \mathrm{Ker}\,(qe_j)$. So, we can define $\alpha: \mathrm{Im}\,(pr) \to \mathrm{Im}\,(qe_j)$ by $((x)pr)\alpha = (x)qe_j$ for all $x \in M_i$. Since M_j is quasi-injective, α can be extended to $s: M_j \to M_j$. Then, obviously, $rs \in e_kSe_j$ and $qe_j = p(rs) \in pS$, so $q \in pS$. Therefore, $e_iN = pS = pe_kS$ is quasi-primitive.

For the quasi-primitivity of ${}_SNe_i$, take $[S(M_i), M_i] \in \mathfrak{F}^f$. Let $[B, M_l] \in \mathfrak{F}^f$ be large next to $[S(M_i), M_i]$ in the linearly ordered component of $\mathfrak{F}^f$, and $t: M_l \to M_i$ such that $(B)t = S(M_i)$. It suffices to show that for a given $0 \neq u = e_jue_i \in e_jNe_i$, there exists an element $a \in S$ with $at = u$ (then $Ne_i = St = Se_lt$). Put $C = (S(M_i))u^{-1}$ i.e. the inverse image of $S(M_i)$ by $u: M_j \to M_i$. Then since u is non-zero, $(C)u = S(M_i)$, so $[C, M_j] \succ [S(M_i), M_i]$. By the choice of $[B, M_l]$, we have $[C, M_j] \succcurlyeq [B, M_l]$. Let $v: M_j \to M_l$ be such that $(C)v = B$. Then $0 \neq (C)vt = (C)u (= S(M_i))$, hence $\mathrm{Im}\,(vt) = \mathrm{Im}\,(u)$. So, $\beta: M_j/\mathrm{Ker}\,(u) \to M_j/\mathrm{Ker}\,(vt)$ with $(\bar{x})\beta = \bar{y}$ if $(x)u = (y)vt$ is well defined. By the quasi-projectivity of M_j, we can lift β to $w: M_j \to M_j$. Then, as easily checked, $(wv)t = u$. So wv is a required element.

a)⇒b) Suppose that an iso-subquotient $U = A/B$ and a pair-subquotient V of M_i are joined, and let $\alpha: M_j \to U = A/B$ be an isomorphism. If V is a factor module M_i/C and $\beta: M_i/C \to M_k$ is a monomorphism, then $M_i \supsetneqq A \supsetneqq C \supseteq B \neq 0$ since A/B and M_i/C are joined. Let $e_iSe_k \ni p: M_i \to M_k$ and $e_jSe_k \ni q: M_j \to M_k$ be as follows.

$$\underbrace{M_i \longrightarrow M_i/C \xrightarrow{\beta} M_k}_{p}, \qquad \underbrace{M_j \xrightarrow{\alpha} A/B \longrightarrow A/C \longrightarrow M_i/C \xrightarrow{\beta} M_k}_{q}.$$

Since ${}_SSe_k$ is serial, there must exist $e_iSe_j \ni r: M_i \to M_j$ with $rq = p$ or $e_jSe_i \ni s: M_j \to M_i$ with $sp = q$. But, this is impossible, since $|\mathrm{Im}\,(p)| = |M_i/C| > |A/C| = |\mathrm{Im}\,(q)|$, $|\mathrm{Ker}\,(p)| = |C| > |C/B| = |\mathrm{Ker}\,(q)|$ and $q \neq 0$.

Next, if V is a submodule of M_i and $\gamma: M_l \to V$ is an epimorphism, then $M_i \supsetneqq A \supseteq V \supsetneqq B \neq 0$. Let $t: M_l \to M_i$ and $u: M_l \to M_j$ be as follows.

$$\underbrace{M_l \xrightarrow{\gamma} V \longrightarrow M_i}_{t}, \qquad \underbrace{M_l \xrightarrow{\gamma} V \longrightarrow V/B \longrightarrow A/B \xrightarrow{\alpha^{-1}} M_j}_{u}.$$

Then, similarly to the above, we conclude that there must exist $v: M_j \to M_i$ with $uv = t$ or $w: M_i \to M_j$ with $tw = u$. But, $|\mathrm{Im}\,(t)| = |V| > |V/B|$

$=|\mathrm{Im}\,(u)|$, $|\mathrm{CoIm}\,(t)|=|M_i/V|>|A/V|=|\mathrm{CoIm}\,(u)|$ and $u\neq 0$. Thus, also in this case, we have a contradiction.

To prove Theorem 2, we need the following lemma.

Lemma. *Let $M=M_1\oplus\cdots\oplus M_n$ be a left R-module and $S=\mathrm{End}\,({}_RM)$. If each M_i is serial and injective (resp. projective), then S is right (resp. left) serial.*

Proof. We prove only the part that the injectivity of each M_i leads to the right seriality of S, because the other part can be proved by quite dual argument. Let $e_i: M\to M_i$ be the projection onto M_i. We show that for $e_iS\ni s: M_i\to M$ and $e_iS\ni t: M_i\to M$ with $\mathrm{Ker}\,(s)\subseteq\mathrm{Ker}\,(t)$, there exists $S\ni u: M\to M$ with $su=t$. Since M_i is serial, $\mathrm{Ker}\,(se_k)=\bigcap_j \mathrm{Ker}\,(se_j)=\mathrm{Ker}\,(s)$ for some index $1\leq k\leq n$, then $\mathrm{Ker}\,(se_k)=\mathrm{Ker}\,(s)\subseteq\mathrm{Ker}\,(t)=\bigcap_j\mathrm{Ker}\,(te_j)\subseteq\mathrm{Ker}\,(te_j)$. So, let $\alpha: M_i/\mathrm{Ker}\,(se_k)\to M_i/\mathrm{Ker}\,(te_j)$ be the canonical epimorphism and $\overline{se}_k: M_i/\mathrm{Ker}\,(se_k)\to\mathrm{Im}\,(se_k)$ and $\overline{te}_j: M_i/\mathrm{Ker}\,(te_j)\to\mathrm{Im}\,(te_j)$ the isomorphisms induced by se_k and te_j, respectively. Consider the commutative diagram

$$\begin{array}{ccccccc} & \nearrow & M_i/\mathrm{Ker}\,(se_k) & \xrightarrow{\overline{se}_k} & \mathrm{Im}\,(se_k) & \longrightarrow & M_k \\ M_i & & \downarrow\alpha & & \downarrow\beta & & \downarrow u_j \\ & \searrow & M_i/\mathrm{Ker}\,(te_j) & \xrightarrow{\overline{te}_j} & \mathrm{Im}\,(te_j) & \longrightarrow & M_j, \end{array}$$

where β exists since both $\overline{se}_k$ and $\overline{te}_j$ are isomorphisms, and u_j exists since M_j is injective. Thus, we have $se_ku_j=te_j$. Taking u_j for each $1\leq j\leq n$, we obtain $t=\sum_j te_j=\sum_j se_ku_j=s\left(\sum_j e_ku_j\right)$ and $\sum_j e_ku_j\in S$, as required.

Now, let L_1 and L_2 be given subideals of e_iS. If there exists $L_1\ni p: M_i\to M$ such that $\mathrm{Ker}\,(p)\subseteq\mathrm{Ker}\,(q)$ for any $L_2\ni q: M_i\to M$, then $L_1\ni pr=q$ for some $r\in S$, so $L_1\supseteq L_2$. If there does not exist such p, then for any $u\in L_1$ we can choose $v\in L_2$ with $\mathrm{Ker}\,(u)\supseteq\mathrm{Ker}\,(v)$, thus $u=vw\in L_2$ for some $w\in S$, so $L_1\subseteq L_2$. This implies that e_iS_S is serial, so S is right serial.

As easily seen from the above proof, it is to be noted that in Lemma the finiteness of the chain that all submodules of M_i and ${}_SSe_i$ (e_iS_S) form is not necessary.

Proof of Theorem 2. Let ${}_RRe$ be a minimal faithful left R-module, then $Q=\mathrm{End}\,(Re_{eRe})$ (c.f. [3, p. 47]). Put ${}_{Q,R}M_S={}_{Q,R}Re_{eRe}$, and let $S\ni 1=\sum_{i=1}^m e_i$ be decomposition of 1 into orthogonal primitive idempotents. Since ${}_RRe$ is a direct sum of injective and projective serial modules, $S=\mathrm{End}\,({}_RRe)$ is serial by the above lemma. We note that M_S is finitely generated (c.f. [3, p. 59]), so let $M_S=M_S^{(1)}\oplus\cdots\oplus M_S^{(n)}$ be a direct sum decomposition into indecomposable modules. Now, suppose that $M_S^{(i)}$ appears as a proper subquotient of $M_S^{(j)}$, that is, $M^{(i)} \simeq A/B$ and $M^{(j)} \supsetneqq A \supseteq B \neq 0$. Let $T(M^{(i)}) \simeq T(A) \simeq T(e_k S)$, and take $x = xe_k \in M^{(i)} \setminus M^{(i)}N$ and $y = ye_k \in A \setminus AN$, where N the radical of S. Since ${}_Q Me_k$ is an indecomposable direct summand of ${}_Q M$, ${}_Q Me_k$ is serial. From $x, y \in Me_k$ it follows there exists $p \in Q$ with $px = y$ or $q \in Q$ with $qy = x$. Since $Q = \mathrm{End}(M_S)$, this implies that there exists $\alpha : M_S^{(i)} \to M_S^{(j)}$ with $\alpha(x) = y$ or $\beta : M_S^{(j)} \to M_S^{(i)}$ with $\beta(y) = x$. But, this is impossible. Therefore, according to Corollary 1, we conclude Q is serial.

The following is a consequence of Theorem 2.

Corollary 2. *If R is a left serial* QF-3 *ring which is a maximal quotient ring, then R is serial.*

References

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