110. A Note on a Characterization of Principal Ideal Domain

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Let D denote a unique factorization domain (UFD) and let K denote its quotient field. In [1] Iwamoto investigated the D-submodules of K where D was given an additional property. This property was stated in [1] as "every principal ideal of D is maximal" which is clearly a misprint. However, if this property is stated as "every principal prime ideal sf D is maximal" then it is easy to see that D is a principal ideal domain (PID) and that, with this property, all of the proofs leading to the description of the D-submodules of K in [1] are correct. In this note it will be shown that the description of the D-submodules of K given in [1] actually characterizes principal ideal domains and so no more general property than PID can be used in [1].

Let f denote a mapping from P, the set of all prime elements of D, into $Z \cup \{-\infty\}$ such that $f(p)>0$ for only a finite number of elements $p \in P$ and let F denote the set of all such mappings. If we let $M(f) =\{x \in K \mid V_p(x) \geq f(p) \text{ for all } p \in P\}$ where V_p is the p-valuation on K then it is easy to see that $M(f)$ is a D-module for all $f \in F$. In [1] it is shown, in view of the comments above, that if D is a PID, then every D-submodule of K is of the form $M(f)$ for some $f \in F$.

Theorem. *Let D denote a UFD. Every D-submodule of K is of the form $M(f)$ for some $f \in F$ if and only if D is a PID.*

Proof. The "if" direction was proved in [1]. Suppose that every D-submodule of K is of the form $M(f)$ for some $f \in F$. Let p_1 and p_2 be two prime elements in D (if there are fewer than two primes in D, the theorem is obviously true). Consider $N=\{d_1/p_1+d_2/p_2 \mid d_1, d_2 \in D\}$. Clearly N is a D-submodule of K. Then, by assumption, $N=M(f)$ for some $f \in F$. Since $1/p_1$ is an element of N, $f(p_1) \leq -1$, and similarly $f(p_2) \leq -1$. Also, since $1 \in N$, $f(p) \leq 0$ for all primes p. This implies that $1/p_1p_2 \in N$. Therefore, $1/p_1p_2=d_1/p_1+d_2/p_2$ for some d_1 and d_2 in D. Consequently, $1=d_1p_2+d_2p_1$ and so p_1 and p_2 are not in the same maximal ideal. Hence every maximal ideal of D contains exactly one prime element which implies that D is a PID.

Note that the proof of the theorem shows that only those D-submodules of K containing D need be considered. Hence a UFD D

is a PID if and only if all of the D-submodules of K containing D are of the form $M(f)$ for some $f \in F$.

Reference

[1] T. Iwamoto: A characterization of submodules of the quotient field of a domain. Proc. Japan Acad., **49**, 140–144 (1973).