174.   On a Density Theorem of Linnik

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(Comm. by Kunihiko KODAIRA, M. J. A., Dec. 12, 1975)

1. Let $N(\alpha, T, \chi)$ be the number of zeros of Dirichlet's L-function $L(s, \chi)$ in the region $\alpha \leq \sigma \leq 1$, $|t| \leq T$, $s = \sigma + it$. Then, recently Montgomery and Selberg [5] (see also [2; p. 40]) have obtained the estimates

$$\sum_{\chi \pmod q} N(\alpha, T, \chi) \ll_\varepsilon (qT)^{(3+\varepsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod q}^* N(\alpha, T, \chi) \ll_\varepsilon (Q^5 T^3)^{(1+\varepsilon)(1-\alpha)}.$$

In their proof a refinement of the large sieve inequality [2; Théorème 7A] as well as the remarkable identity of Lemma 1 below (both due to Selberg) play vital roles. Also the other novelty of their method lies in that they have dispenced with the Power Sum Method of Turán which was essential in the former results due to Fogels and Gallagher in this field.

The purpose of this note is to announce a further improvement:

Theorem 1. *If $4/5 \leq \alpha \leq 1$, then we have*

$$\sum_{\chi \pmod q} N(\alpha, T, \chi) \ll_\varepsilon (q^2 T^3)^{(1+\varepsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod q}^* N(\alpha, T, \chi) \ll_\varepsilon (Q^4 T^3)^{(1+\varepsilon)(1-\alpha)}.$$

As for the proof we mention only that this is deduced by combining the lemmas of the next paragraph with the zero-detecting devices of [4; pp. 104–110], and that we actually obtain a result in which the exponent $1 + \varepsilon$ is replaced by $1/\alpha + \varepsilon$ in the above, then it can be expressed as in our theorem by the recent work of Jutila [3].

In the proof of the above result (i.e. in Lemma 2 below) we have used the Pólya-Vinogradov theorem, and if we use Burgess' result instead of it, then we can prove

Theorem 2. *In the vicinity of $\alpha = 1$, we have*

$$\sum_{\chi \pmod q} N(\alpha, T, \chi) \ll_\varepsilon (q^{9/8} T^6)^{(1+\varepsilon)(1-\alpha)},$$

$$\sum_{q \leq Q} \sum_{\chi \pmod q}^* N(\alpha, T, \chi) \ll_\varepsilon (Q^{9/4} T^6)^{(1+\varepsilon)(1-\alpha)}.$$

The proof of the results of this note will be published elsewhere.

2. Our main lemmas are as follows:

Lemma 1. *Let r be a square-free integer and let $\psi_r(n) = \mu((r, n))\varphi((r, n))$. Further let*

$$M(s, \chi, \phi_r) = \sum_{d=1}^{\infty} \frac{\xi_d}{d^s} \chi(d)\phi_r(d) \prod_{p|r/(r,d)} \left(1 - \frac{\chi(p)}{p^{s-1}}\right)$$

with arbitrary complex numbers $\xi_d = O(1)$. Then we have, for $\mathrm{Re}\,(s) > 1$,

$$L(s, \chi)M(s, \chi, \phi_r) = \sum_{n=1}^{\infty} \frac{\chi(n)\phi_r(n)}{n^s} \left(\sum_{d|n} \xi_d\right).$$

Lemma 2. *Let* r, r' *be square-free and let* $\delta_{r,r'}$ *be the Kronecker symbol. Let* λ *be a Dirichlet character* (mod q). *Further let*

$$E(M, N\,;\,\chi\phi_r\phi_{r'}) = \sum_{M < n \le M+N} \chi(n)\phi_r(n)\phi_{r'}(n).$$

Then, if χ *is non-principal, we have, for any* M *and* $N > 0$ *and for any* r, r',

$$E(M, N\,;\,\chi\phi_r\phi_{r'}) \ll q^{1/2} (\log q) \prod_{p|r} (p+1) \prod_{p|r'} (p+1).$$

Also, if χ *is principal, we have*

$$E(M, N\,;\,\chi\phi_r\phi_{r'}) = \frac{\varphi(qr)}{q} N\delta_{r,r'} + O\left\{q^{1/2} (\log q) \prod_{p|r} (p+1) \prod_{p|r'} (p+1)\right\},$$

provided $(rr', q) = 1$.

Lemma 3. *Let* χ_j (mod q_j) *be mutually distinct primitive characters such that* $q_j \le Q$. *Then we have, for any* $J, R, M, N \ge 1$ *and for any complex numbers* $c(n)$,

$$\sum_{j \le J} \sum_{\substack{r \le R \\ (r, q_j)=1}} \frac{q_j \mu^2(r)}{\varphi(q_j r)} \left| \sum_{M < n \le M+N} c(n)\chi_j(n)\phi_r(n) \right|^2$$
$$\le \{N + O(JQR^2 (\log QR)^2)\} \sum_{M < n \le M+N} |c(n)|^2.$$

Further, if χ_j *runs over characters* (mod q) *only, then in the above estimate* Q *should be replaced by* $q^{1/2}$ *as well as* χ_j *are not restricted to be primitive.*

Lemma 4. *Let* $s_{j,k} = \sigma_{j,k} + it_{j,k}$ *be complex numbers such that*

$$\operatorname*{Min}_{j,k} \sigma_{j,k} = \sigma, \qquad \operatorname*{Max}_{j,k} |t_{j,k}| \le T,$$
$$\operatorname*{Min}_{j} \operatorname*{Min}_{1 \le k < k' \le K_j} |t_{j,k} - t_{j,k'}| = \delta,$$

where $T \ge 1$ *and* $\delta > 0$. *Then, under the same conventions as in the preceding lemma, we have, for any* K_j *and* $A > 1$,

$$\sum_{j \le J} \sum_{\substack{r \le R \\ (r, q_j)=1}} \frac{q_j \mu^2(r)}{\varphi(q_j r)} \sum_{k \le K_j} \left| \sum_{N < n \le N^A} c(n)\chi_j(n)\phi_r(n) n^{-s_{j,k}} \right|^2$$
$$\ll_A (\delta^{-1} + \log N) \sum_{N < n \le N^A} (n + JQTR^2 (\log QR)^2) |c(n)|^2 n^{-2\sigma}.$$

Lemma 5 (Barban-Vehov [1], Motohashi [6]). *Let* $\lambda_d = \mu(d)$ *if* $d \le z$, $= \mu(d) (\log z^{1+\epsilon}/d)/(\log z^\epsilon)$ *if* $z < d \le z^{1+\epsilon}$, *and* $= 0$ *if* $z^{1+\epsilon} \le d$. *Then we have, for any* $x > 1$,

$$\sum_{1 < n \le x} \left(\sum_{d|n} \lambda_d\right)^2 \ll_\epsilon x/(\log z).$$

References

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