173. Weight Functions of the Class (A_∞) and Quasi-conformal Mappings

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§1. **Introduction.** In the following we use G as an open subset of R^n, Q (or P) as a cube with sides parallel to coordinates axis, E as a measurable set and $\chi(E)$ as the characteristic function of E. When f is a measurable function defined on R^n, $\sup\left\{\left(|Q|^{-1}\int_Q |f(y)|^p dy\right)^{1/p} \mid Q\ni x\right\}$ will be denoted by $M_p(f)(x)$. If $\varphi: G_1\to G_2$ is totally differentiable at x, the Jacobian matrix of φ at x will be denoted by $\Phi(x)$ and $|\det\Phi(x)|$ by $J_\varphi(x)$. For ACL (absolutely continuous on lines) and BMO (bounded mean oscillation) see Reimann [4].

In Reimann [4] he proved the following theorem.

Theorem A. *Let φ be a homeomorphism of R^n onto itself, ACL and totally differentiable a.e. and assume that $|\varphi(\cdot)|$ and $|\varphi^{-1}(\cdot)|$ are absolutely continuous set functions in R^n. Then φ is quasiconformal iff there exists $C>0$ such that $\|f\circ\varphi^{-1}\|_*\leqslant C\|f\|_*$ for any BMO function f, where $\|\cdot\|_*$ means the BMO norm.*

Using his idea, some other characterizations of quasiconformal mappings are possible. Theorem 1 and Corollary 1 are characterizations by Hardy-Littlewoods' maximal functions and Theorem 2 is a characterization by some kind of measures.

§2. **The Hardy-Littlewoods' maximal functions and quasiconformal mappings**

Theorem 1. *Let φ be a homeomorphism of G_1 onto G_2, ACL and totally differentiable a.e. Then the followings are equivalent.*

(I) *φ is a quasiconformal mapping.*

(II) *There exist $C>0$ and $\infty>p>1$ satisfying the following conditions:*

For $\forall x\in G_1$ there exists $r(x)>0$ such that

$$\begin{aligned}&\sup\left\{|Q|^{-1}\int_Q f(y)dy \mid \operatorname{diam} Q<r(x),\ Q\ni x\right\}\\ &\quad\leqslant C\sup\left\{\left(|Q|^{-1}\int_Q (f\circ\varphi^{-1}(y))^p dy\right)^{1/p} \mid Q\ni\varphi(x),\ Q\subset G_2\right\},\end{aligned}\tag{1}$$

$$\begin{aligned}&\sup\left\{|Q|^{-1}\int_Q f\circ\varphi^{-1}(y)dy \mid \operatorname{diam} Q<r(x),\ Q\ni\varphi(x)\right\}\\ &\quad\leqslant C\sup\left\{\left(|Q|^{-1}\int_Q f(y)^p dy\right)^{1/p} \mid Q\ni x,\ Q\subset G_1\right\}\end{aligned}\tag{2}$$

for any nonnegative measurable function f and

$$\{y\,|\,|y-x|<r(x)\}\subset G_1,\qquad \{y\,|\,|y-\varphi(x)|<r(x)\}\subset G_2.$$

Corollary 1. *Let φ be a homeomorphism of R^n onto itself, ACL and totally differentiable a.e. Then φ is quasiconformal iff there exist $1<p_1<p_2<\infty$, $C_1>0$, $C_2>0$ such that*

$$M_1(f)(x)\leqslant C_1M_{p_1}(f\circ\varphi^{-1})(\varphi(x))\leqslant C_2M_{p_2}(f)(x)$$

for any measurable function f defined on R^n and for any $x\in R^n$.

Proof of Theorem 1. (I)→(II). From Gehring [2] Lemmas 3 and 4, there exist $\varepsilon>0$ and $C>0$ such that

$$\left(|Q|^{-1}\int_Q J_\varphi(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)}\leqslant C\,|Q|^{-1}\int_Q J_\varphi(x)dx$$

for any cube Q with diam $\varphi(Q)\leqslant$ dist $(\varphi(Q),\partial G_2)$. Then from Coifman and Fefferman [1] Theorem 5, there exist $C>0$ and $\infty>p>1$ such that

$$|Q|^{-1}\int_Q J_\varphi(x)dx\leqslant C\left(|Q|^{-1}\int_Q J_\varphi(x)^{-p'/p}dx\right)^{-p/p'}\qquad(3)$$

for any cube Q with diam $\varphi(Q)\leqslant$ dist $(\varphi(Q),\partial G_2)$, where $1/p+1/p'=1$. Therefore,

$$\begin{aligned}|Q|^{-1}\int_Q f(y)dy&\leqslant|Q|^{-1}\left(\int_Q J_\varphi(y)^{-p'/p}dy\right)^{1/p'}\left(\int_Q f(y)^pJ_\varphi(y)dy\right)^{1/p}\\&\leqslant C\left(\int_Q J_\varphi(y)dy\right)^{-1/p}\left(\int_Q f(y)^pJ_\varphi(y)dy\right)^{1/p}\\&=C\left(|\varphi(Q)|^{-1}\int_{\varphi(Q)}(f\circ\varphi^{-1}(y))^pdy\right)^{1/p}.\end{aligned}$$

But if diam $\varphi(Q)/$dist $(\varphi(Q),\partial G_2)$ is sufficiently small, there exists a cube P such that $\varphi(Q)\subset P\subset G_2$ and $|P|\leqslant C\,|\varphi(Q)|$, where C depends only on φ [see Gehring [2] Lemma 4]. This proves (1). Since φ^{-1} is also quasiconformal [see Mostow [3] Theorem 9.3], (2) can be proved similarly.

(II)→(I). The proof of Theorem 3 in Reimann [4] can be used as it stands, but in our case we can prove by means of a simpler function. From (II), $|\varphi(\cdot)|$ and $|\varphi^{-1}(\cdot)|$ are absolutely continuous set functions, so by the same argument as Reimann [4] Theorem 3, it surfaces to prove that there exists $C>0$ satisfying

$$\sup\{|\Phi(x_0)\xi|^n\,|\,|\xi|=1,\ \xi\in R^n\}\leqslant CJ_\varphi(x_0)$$

for any $x_0\in R^n$ where φ is differentiable and $J_\varphi(x_0)\neq0$. For this end we have only to prove $\lambda_n\leqslant C'$ where C' is independent of x_0 and

$$\Phi(x_0)=\lambda\rho\begin{bmatrix}1&&&0\\&\lambda_2&&\\&&\ddots&\\0&&&\lambda_n\end{bmatrix}\sigma,\quad \rho,\sigma\in 0(n),\quad 1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n.$$

Let $g(x)$ be $\chi([1,2]\times[-1/2,1/2]\times\cdots\times[-1/2,1/2])(x)$ and $f_\varepsilon(x)$ be $g(\varepsilon^{-1}\lambda^{-1}\rho^{-1}(\varphi(x)-\varphi(x_0)))$. Then replacing f by f_ε, the right hand side of (2) tends to

$$CM_p(\chi([1,2]\times[-2^{-1}\lambda_2^{-1},2^{-1}\lambda_2^{-1}]\times\cdots\times[-2^{-1}\lambda_n^{-1},2^{-1}\lambda_n^{-1}]))(0)$$

as ε converges to 0. On the other hand, the left hand side of (2) tends to $M_1 g(0)$. So, $C < \lambda_n^{-1/p}$, i.e. $\lambda_n \leqslant C$.

Proof of Corollary 1. When $G_1 = G_2 = R^n$, we can take $r(x) \equiv \infty$.

§ 3. (A$_\infty$)-measures and quasiconformal mappings. Coifman and Fefferman [1] proved the following theorem.

Theorem B. *When μ is a measure defined on the Borel sets of R^n, the followings are equivalent.*

(B−I) *There exist $\delta_1 > 0$ and $C_1 > 0$ such that*

$$\mu(E)/\mu(Q) \leqslant C_1(|E|/|Q|)^{\delta_1} \qquad \text{for } \forall E \subset \forall Q.$$

(B−II) *There exist $\delta_2 > 0$ and $C_2 > 0$ such that*

$$|E|/|Q| \leqslant C_2(\mu(E)/\mu(Q))^{\delta_2} \qquad \text{for } \forall E \subset \forall Q.$$

(B−III) *$d\mu = w(x)dx$ and there exist $C > 0$ and $a > 0$ such that*

$$|Q|^{-1} \int_Q w(x)dx \leqslant C\left(|Q|^{-1} \int_Q w(x)^{-a} dx\right)^{-1/a} \qquad \text{for } \forall Q.$$

Definition. The class of w (or μ) which satisfies B−I, II, III is called (A_∞).

For the relation between (A_∞) and BMO, Reimann [4] proved the following result.

Theorem C. *We define $\sim$ and $\approx$ as follows.*

$$f \sim g \text{ iff } \exists a > 0, \exists b \in R \text{ s.t. } f = ag + b$$
$$u \approx v \text{ iff } \exists a, b > 0 \text{ s.t. } u = av^b.$$

Then $w \mapsto \log w$ defines a one-to-one mapping from $A_\infty/\approx$ onto $BMO/\sim$.

Using Theorem C, we can prove the following theorem.

Theorem 2. *Under the same condition as in Corollary 1 φ is quasiconformal iff*

$$\mu(\varphi^{-1}(\cdot)), \mu(\varphi(\cdot)) \in (A_\infty) \qquad \text{for } \forall \mu \in (A_\infty).$$

Proof $(\rightarrow)$. Let Q be any cube in R^n, then there exists a cube $P \supset \varphi(Q)$ such that $|P| \leqslant C|\varphi(Q)|$, where C is independent of Q [see Gehring [2] Lemma 4]. From (3), $|\varphi(\cdot)| \in (A_\infty)$. Then for $\forall \mu \in (A_\infty)$, $\forall E \subset Q$

$$\begin{aligned}\mu(\varphi(E))/\mu(\varphi(Q)) &\leqslant C\mu(\varphi(E))/\mu(P) \\ &\leqslant C(|\varphi(E)|/|P|)^{\delta_1} \leqslant C(|\varphi(E)|/|\varphi(Q)|)^{\delta_1} \\ &\leqslant C(|E|/|Q|)^{\delta_2}.\end{aligned}$$

So, $\mu(\varphi(\cdot)) \in (A_\infty)$. Since φ^{-1} is quasiconformal, $\mu(\varphi^{-1}(\cdot))$ also belongs to (A_∞).

$(\leftarrow)$. From the fact $dx \in (A_\infty)$ and the hypothesis, $|\varphi(\cdot)|$ and $|\varphi^{-1}(\cdot)|$ belong to (A_∞), i.e. $J_\varphi(x), J_{\varphi^{-1}}(x) \in (A_\infty)$. Let f be any element of $BMO(R^n)$. Then from Theorem C there exists $\varepsilon > 0$ such that $e^{\varepsilon f(x)} \in (A_\infty)$. Then from the hypothesis the set function $E \mapsto \int_{\varphi^{-1}(E)} e^{\varepsilon f(x)} dx$ belongs to (A_∞), i.e. $e^{\varepsilon f \circ \varphi^{-1}} J_{\varphi^{-1}}(x) \in (A_\infty)$. From Theorem C $\varepsilon f \circ \varphi^{-1} + \log J_{\varphi^{-1}} \in BMO$ so $f \circ \varphi^{-1} \in BMO$. Then by the closed graph theorem $(\leftarrow)$ part is proved.

References

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