169. *Approximation Theorem on Stochastic Stability*

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§ 1. This paper treats the approximation theorem on the stability theory of dynamical systems given by stochastic differential equations. Consider a dynamical system in $\boldsymbol{R}^n$:

$$(1)\qquad dx_i(t)=\sum_{k=1}^{n}\sigma_{ik}(x(t))dB_k(t)+b_i(x(t))dt \qquad (i=1,\cdots,n)$$

(in this paper, we always assume that coefficients of (1) are Lipschitz continuous). If we assume that for $m\geqq 1$

$$(2)\qquad \begin{cases}\sigma_{ik}(x)=\tilde{\sigma}_{ik}(\lambda)\,|x|^m+o(|x|^m)\\ b_i(x)=\tilde{b}_i(\lambda)\,|x|^{2m-1}+o(|x|^{2m-1}) \qquad |x|\to 0,\end{cases}$$

where $\lambda=x/|x|$, then the first approximation of (1) is defined by

$$(3)\qquad dx_i(t)=\sum_k \tilde{\sigma}_{ik}(\lambda(t))\,|x(t)|^m\,dB_k(t)+\tilde{b}_i(\lambda(t))\,|x(t)|^{2m-1}\,dt.$$

Following to Khas'minskii [2], we call $x(t)$ asymptotic stable in probability if $\lim_{|x|\to 0} P_x\{\lim_{t\to\infty}|x(t)|=0\}=1$, asymptotic unstable in probability if $P_x\{\lim_{t\to\infty}|x(t)|=\infty\}=1$ for all x $(\neq\{0\})$, divergent in probability if $P_x\{\sup_{t>0}|x(t)|>\varepsilon\}=1$ for all x $(\neq\{0\})$ and small $\varepsilon>0$.

The main theorems are:

Theorem 1. *If the solution of* (3) *is asymptotic stable in probability, then that of* (1) *is so.*

Theorem 2. *If the solution of* (3) *is asymptotic unstable in probability, then that of* (1) *is divergent in probability.*

When $m=1$, the results have been already proved by Khas'minskii [2] and Pinsky [4].

In § 2 we sketch proofs of Theorems 1 and 2. In § 3 they are applied to a limit behaviour of a stochastic process on a two dimensional compact manifold, which is useful for studying the stability of three dimensional linear systems (see [1]).

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§ 2. **Remark 1.** In this section it will be proved that the stability of (3) is equivalent to that of

$$(4)\qquad dx_i(t)=\sum_k \tilde{\sigma}_{ik}(\lambda(t))\,|x(t)|\,dB_k(t)+\tilde{b}_i(\lambda(t))\,|x(t)|\,dt.$$

Thus, a little modification of Khas'minskii's sharp stability criterion formulated in [1] is applicable to (3).

Remark 2. If $\sigma_{ik}(\lambda)\equiv 0$ and $b_i(\lambda)\equiv 0$ in (3), then the solution of (3) is not asymptotic stable in probability nor asymptotic unstable.

Outline of proofs of Theorems 1 and 2. Define $T(t) =\int_0^t |x(u)|^{2m-2}\,du$. Since $P_x\{0<|x(t)|<\infty,\ t\geqq 0\}=1$ (see [2]), $T^{-1}(t)$ is well defined. By the time substitution $T^{-1}(t)$, (1) and (3) are respectively transformed into

$$(5)\qquad \begin{aligned} dx_i(T^{-1}(t))&=\sum_k \frac{\sigma_{ik}(x(T^{-1}(t)))}{|x(T^{-1}(t))|^{m-1}}d\tilde{B}_k(t)+\frac{b_i(x(T^{-1}(t)))}{|x(T^{-1}(t))|^{2m-2}}dt,\\ dx_i(T^{-1}(t))&=\sum_k \tilde{\sigma}_{ik}(\lambda(T^{-1}(t)))\,|x(T^{-1}(t))|\,d\tilde{B}_k(t)\\ &\quad+\tilde{b}_i(\lambda(T^{-1}(t)))\,|x(T^{-1}(t))|\,dt, \end{aligned}$$

where $\tilde{B}(t)$ are suitable Brownian motions. Now the theorems follow from a slight modification of Theorems 7.1.1 and 7.2.3 in [2].

Especially, if coefficients of (1) are C^∞-class in some neighbourhood of $\{0\}$, and if $\sigma_{ij}(0)=0$ and $b_i(0)=0$, then they are expanded as

$$\begin{aligned} \sigma_{ij}(x)&=\sum_{k_1}\sigma_{ijk_1}x_{k_1}+\sum_{k_1,k_2}\sigma_{ijk_1k_2}x_{k_1}x_{k_2}+\cdots,\\ b_i(x)&=\sum_{k_1}b_{ik_1}x_{k_1}+\sum_{k_1,k_2}b_{ik_1k_2}x_{k_1}x_{k_2}+\cdots. \end{aligned}$$

Let $M_\sigma=\min\{s:\ \max_{i,j,k_1,\cdots,k_s}|\sigma_{ijk_1\cdots k_s}|>0\}$, $M_b=\min\{s:\ \max_{i,k_1,\cdots,k_s}|b_{ik_1\cdots k_s}|>0\}$, and $L=\min\left\{\dfrac{M_b+1}{2},\ M_\sigma\right\}$. Set

$$(6)\qquad d\hat{x}_i(t)=\sum_j \hat{\sigma}_{ij}(\hat{\lambda}(t))\,|\hat{x}(t)|^L\,dB_j(t)+\hat{b}_i(\hat{\lambda}(t))\,|\hat{x}(t)|^{2L-1}\,dt,$$

where

$$\hat{\sigma}_{ij}(\lambda)=\begin{cases}\displaystyle\sum_{k_1,\cdots,k_L}\frac{\sigma_{ijk_1\cdots k_L}x_{k_1}\cdots x_{k_L}}{|x|^L} & L \text{ is integer}\\ 0 & L \text{ is not integer,}\end{cases}$$

$$\hat{b}_i(\lambda)=\sum_{k_1,\cdots,k_{2L-1}}\frac{b_{ik_1\cdots k_{2L-1}}x_{k_1}\cdots x_{k_{2L-1}}}{|x|^{2L-1}}.$$

Then (2) always holds for coefficients in (1) and (6), with $m=L$.

§ 3. Let M be a two dimensional, compact, analytic manifold. A diffusion process $\pi(t)$ on M is given by the stochastic differential equations, defined on each local chart (U_α, Ψ_α) (see [5]),

$$(7)\qquad \Psi_\alpha(\pi(t))=\Psi_\alpha(\pi(s))+\int_s^t a_\alpha(\Psi_\alpha(\pi(u)))dB(u)+\int_s^t b_\alpha(\Psi_\alpha(\pi(u)))du.$$

Assume that the coefficients $\{a_\alpha, b_\alpha\}$ in (7) are C^∞-class. (For stochastic differential equations induced by Khas'minskii's sharp stability criterion, this condition always holds and M is the unit spherical surface, see [1] or [4].)

Let a point q_0 on M be such that

$$(8)\qquad \|(a_\alpha a_\alpha^*)(\Psi_\alpha(q_0))\|=0 \quad\text{and}\quad |b_\alpha(\Psi_\alpha(q_0))|=0.$$

For simplicity, let $\Psi_\alpha(q_0)=\{0\}$. If $x(t)\equiv\Psi(\pi(t))$, then the approximation of (7) is given by (6) in a neighbourhood of $\{0\}$. By Remark 1, we may assume that $L=1$ in (6). After extending naturally (6) to the whole space R^2, we have (see [1])

$$(9)\qquad \lim_{t\to\infty}\frac{1}{t}\log|\hat{x}(t)|=\lim_{t\to\infty}\frac{1}{t}\int_0^t Q(\hat{\lambda}(u))du \qquad \text{a.s.},$$

where $\hat{\lambda}(t)=\hat{x}(t)/|\hat{x}(t)|=(\cos\hat{\theta}(t), \sin\hat{\theta}(t))$ and Q is a function obtained by Ito's formula.

Following to [3], we can really compute the right hand side of (9), which we denote by J_{q_0}. In general, J_{q_0} is depending on a starting point of $\hat{x}(t)$ and random. However if we assume that

$$(10)\qquad \|(\hat{\sigma}\hat{\sigma}^*)(\lambda)\|>0 \qquad \text{for any } \lambda,$$

then J_{q_0} is a constant (see [3]).

From Theorems 1 and 2, we see that $\pi(t)$ is asymptotic stable (divergent) in probability at q_0 if $J_{q_0}<0$ (>0). From those and the other results formulated in [2], we have:

Theorem 3. *Let q_i $(i=1,\cdots,m)$ be such points as* (8) *and* (10) *hold. Let rank* $[(aa^*)(\Psi(q))]=2$ *for all q $(\neq q_i$'s).*

(i) *If $J_{q_i}>0$ for $1\leqq i\leqq m$, then $\pi(t)$ is recurrent on $M-\{q_i: i=1,\cdots,m\}$, i.e., for any open set $\mathfrak{D}\subset M$,*

$$P_q\{\tau_{\mathfrak{D}}<\infty\}=1 \qquad q\notin\{q_i: i=1,\cdots,m\},$$

where $\tau_{\mathfrak{D}}$ is the first hitting time for $\mathfrak{D}$.

(ii) *If $J_{q_i}<0$ for $1\leqq i\leqq j$ and if $J_{q_i}>0$ for $j+1\leqq i\leqq m$, then*

$$P_q\{\lim_{t\to\infty}\pi(t)\in\{q_i: i=1,\cdots,j\}\}=1$$

for all $q\notin\{q_i: i=j+i,\cdots,m\}$.

References

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