166. On Closed Countably-Compactifications and Quasi-Perfect Mappings

By Takesi ISIWATA
Tokyo Gakugei University

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1975)

Throughout this paper, by a space we shall mean a completely regular T_1-space. According to Morita [14], [15], a space S is a *countably-compactification* $(=c-cf)$ of a given space X if

a) S is countably compact $(=cc)$ and contains X as a dense subset, and

b) every cc closed subset of X is closed also in S. In case X admits a $c-cf$, X is said to be *countably-compactifiable*. Since X is countably-compactifiable if and only if X has a $c-cf$ S with $X\subset S\subset \beta X$ ([14], Proposition 3.4), in the sequel we will consider only a $c-cf$ S of X as a subspace S of βX with the exception of § 3. Interesting results concerning countably-compactifiability have been obtained by Morita. For example, an M-space X is countably-compactifiable if and only if X is homeomorphic to a closed subset of a product space of a countably compact space and a metric space [14], [15]. In [10] we introduced a notion of closed $c-cf$ and investigated some properties and characterizations of spaces with the closed $c-cf$. Let S be a $c-cf$ of X and $X^*=\beta X-X$ and $S^*=S\cap X^*$. S^* is called *the X^*-section of S*. In case S^* is closed in X^*, we say that S is the *closed $c-cf$ of X*. In Theorem 3.5 [10] it is proved that if X admits a closed $c-cf$, then it is uniquely determined.

Concerning relations between countably-compactifiability of given spaces and maps, it is natural to ask whether countably-compactifiability of X (resp. Y) implies one of Y (resp. X) where Y is a quasi-perfect image of X. For this problem, the following results have been obtained.

Theorem A (Morita [14], Proposition 4.2). *Let f be a perfect map from X onto Y. If Y is countably-compactifiable, then so is X.*

Theorem B (Hoshina [2]). *Let f be a quasi-perfect map from X onto Y and X admits a $c-cf$. Then we have*

1) *if either Y is normal or an M-space, then Y admits a $c-cf$.*

2) *if f is open, then Y admits a $c-cf$.*

Theorem A implies that if f is a perfect map from X onto Y with a $c-cf$ T, then $S=(\beta f)^{-1}T=X\cup S^*$ is a $c-cf$ of X and $f_S=\beta f|S$ is obviously a perfect map from S onto T where $S^*=(\beta f)^{-1}T^*$ and βf is the Stone extension of f. But as shown by Example 3.1, S is not necessarily a $c-cf$ of X when f is quasi-perfect. On the other hand, even if f is a perfect map from X with the closed $c-cf$ S onto Y, $\beta f(S)$ is not necessarily a $c-cf$ of Y as shown by Example 3.2 which also shows that "M-ness" is not invariant under a perfect map as proved by Morita [12].

In this paper, concerning these examples and Theorems A and B we shall prove two Theorems 1.5 and 2.1 which imply the following as special cases.

1) If f is a quasi-perfect map from X onto Y with a $c-cf$ T and either X is an M-space or Y is locally compact, then the closedness of f_S is equivalent to the condition that S is a $c-cf$ of X.

2) Under the assumption that f is perfect map from an M-space X onto Y and X admits the closed $c-cf$, the "M-ness" of Y is equivalent to the condition that Y admits the closed $c-cf$.

§ 1. In this section, we assume that for a map f from X onto Y and $Y \subset T \subset \beta Y$, we put

$$T^* = T \cap Y^*, (\beta f)^{-1} T^* = S^*, S = X \cup S^* \quad \text{and} \quad f_S = \beta f \mid S.$$

Verifications of Lemmas 1.1 and 1.3 and Corollary 1.2 are easy from Theorem A and the definitions of $c-cf$ and quasi-perfect maps.

Lemma 1.1. *Let f be quasi-perfect. If S is a $c-cf$ of X, then T is a $c-cf$ of Y.*

Remark. The converse of Lemma 1.1 is not necessarily true even if f is an open quasi-perfect map from an M-space onto an M-space as shown by 4) and 7) of Example 3.1.

Corollary 1.2. *If in Lemma* 1.1, *f is perfect, then S is a $c-cf$ (resp. the closed $c-cf$) of X if and only if T is a $c-cf$ (resp. the closed $c-cf$) of Y.*

Lemma 1.3. *If f is quasi-perfect and T is a $c-cf$ of Y and f_S is closed, then S is a $c-cf$ of X.*

Lemma 1.4. *Let f be quasi-perfect and S be a $c-cf$ of X. If one of the following conditions* i)$\sim$iii) *is satisfied, then f_S is closed.*

i) *Y is locally compact.*

ii) *S is the closed $c-cf$ of X.*

iii) *X is an M-space.*

Proof. We notice that f being quasi-perfect, $f^{-1}(y)$ is cc and hence $\mathrm{cl}_{\beta X} f^{-1}(y) = (\beta f)^{-1}(y) \subset \mu X$[1] and $S^* \cap (\beta f)^{-1}(y) = \phi$ for every $y \in Y$ [7]. Let F be a closed subset of S and $E = \beta f(F)$. We shall prove that E is closed in T. Suppose that there exists a point $q \in (\mathrm{cl}_{\beta Y} E - E) \cap T$.

In case $q \in T^*$. Since $(\beta f)^{-1}(q) \subset S^*$ and F is closed in S. $(\beta f)^{-1}(q) \cap \mathrm{cl}_{\beta X} F = \phi$ and hence there is an open set U of βX such that $(\beta f)^{-1}(q)$

1) μX is the completion of X with respect to its finest uniformity and it is known that the relation: $X \subset \mu X \subset \upsilon X \subset \beta X$ holds (for example, see [13]).

$\subset U$ and $U \cap \mathrm{cl}_{\beta X} F = \emptyset$. $\beta Y - \beta f(\beta X - U)$ is an open set of βY containing q which is disjoint from $\beta f(\mathrm{cl}_{\beta X} F) = \mathrm{cl}_{\beta Y} E$. This implies $q \in \mathrm{cl}_{\beta Y} E$ which is a contradiction. This allows us to assume that $q \in Y$. Let us put

$$F_X = F \cap X, F^* = F \cap S^*, E_Y = E \cap Y \quad \text{and} \quad E^* = E \cap T^*.$$

Since $S^* = (\beta f)^{-1} T^*$, we have $E_Y = f(F_X)$ and $\beta f(F^*) = E^*$.

a) Since f is closed, E_Y is closed in Y and hence $q \notin \mathrm{cl}_{\beta Y} E_Y$ which implies $q \in \mathrm{cl}_{\beta Y} E^*$. (Thus if Y is locally compact, then f_S is a closed map.)

b) In case $q \in Y \cap \mathrm{cl}_{\beta Y} E^*$ and $q \notin \mathrm{cl}_{\beta Y} E_Y$. Since $\beta f(\mathrm{cl}_{\beta X} F^*) = \mathrm{cl}_{\beta Y} E^* \ni q$, we have $(\beta f)^{-1}(q) \cap \mathrm{cl}_{\beta X} F^*$ is not empty and contained in $X^* - S^*$. (Thus, if S is the closed $c-cf$ of X, then f_S is closed.)

c) Now suppose that X is an M-space. Since μX is a paracompact M-space [11], μX is of countable type [1] and hence there is a family $\{U_n\}$ of open sets of βX which is a neighborhood base at K with $\mathrm{cl}_{\beta X} U_{n-1} \subset U_n$ where K is a compact subset of μX containing q. $(\beta f)^{-1}(q)$ being compact and disjoint from $\mathrm{cl}_{\beta X} F_X$, there is an open set G of βX such that

$$\mathrm{cl}_{\beta X} F_X \cap \mathrm{cl}_{\beta X} G = \emptyset \quad \text{and} \quad (\beta f)^{-1}(q) \subset G.$$

Let us put $K_n = \mathrm{cl}_{\beta X} G \cap \mathrm{cl}_{\beta X} U_n \cap F$. Since $\mathrm{cl}_{\beta X} F^* \cap (\beta f)^{-1}(q) \neq \emptyset$, $K_n \neq \emptyset$ and K_n is cc, we have that $K_0 = \cap K_n \neq \emptyset$ and $K_0 \subset S^*$. On the other hand, we have $K_0 \subset \mu X$ which is impossible. Thus if X is an M-space, then f_S is closed.

Remark. T is always a $c-cf$ of Y by Lemma 1.1.

Theorem 1.5. *Let f be a quasi-perfect map from X onto Y.*

1) *If either X is an M-space or Y is locally compact, then S is a $c-cf$ of X if and only if T is a $c-cf$ of Y and f_S is closed.*

2) *S is the closed $c-cf$ of X if and only if T is the closed $c-cf$ of Y and f_S is closed.*

Proof. 1) is an immediate consequence of Lemmas 1.1, 1.2 and 1.4. To prove 2), it is sufficient to show that if S is the closed $c-cf$ of X, then T is also the closed $c-cf$ of Y. S^* being closed in X^*, $\mathrm{cl}_{\beta X} S^*$ is compact and contained in S. We have therefore $\mathrm{cl}_{\beta Y} T^* = (\beta f)(\mathrm{cl}_{\beta X} S^*) \subset \beta f(S) = T$, that is, T is the closed $c-cf$ of Y.

Corollary 1.6. *Let f be a quasi-perfect map from X onto Y with a $c-cf$ T.*

1) *If either X is an M-space or Y is locally compact, then S is a $c-cf$ of X if and only if f_S is closed.*

2) *If T is the closed $c-cf$ of Y, then S is the closed $c-cf$ of X if and only if f_S is closed.*

§ 2. In this section we consider only M-spaces.

Theorem 2.1. *Let f be a perfect map from an M-space X onto*

Y, then Y admits the closed $c-cf$ if and only if 1) *X admits the closed $c-cf$ and* 2) *Y is an M-space.*

Proof. *Necessity.* Let T be the closed $c-cf$ of Y and $S^*=(\beta f)^{-1}T^*$. Then $S=X\cup S^*$ is the closed $c-cf$ of X by Corollary 1.2. Since X is an M-space, $S^*=\beta X-\mu X$ by Theorem 5.2 [10] and μX is a paracompact M-space. Thus $S^*=(\beta f)^{-1}T^*$ implies $(\beta f)^{-1}\mu Y=\mu X$ which shows that μY is a paracompact M-space and hence Y is an M'-space [13]. On the other hand, Y is an M^*-space and countably paracompact [3]. Thus Y is an M-space [6].

Sufficiency. Since X and Y are M-spaces, $\mu f=\beta f\,|\,\mu X$ is a perfect map from μX onto μY by Theorem 3.4 [13]. Thus by Corollary 1.2 Y admits the closed $c-cf$ $T=Y\cup(\beta Y=\mu Y)$.

Corollary 2.2. *If f is a perfect map from an M-space X onto an M-space Y, then X admits the closed $c-cf$ if and only if Y admits the closed $c-cf$.*

Corollary 2.3. *If f is a perfect map from an M-space X onto Y and X admits the closed $c-cf$, then Y is an M-space if and only if Y admits the closed $c-cf$.*

Theorem 2.4. *Let f be an open quasi-perfect map from an M-space X onto Y, then*

1) *If X admits the closed $c-cf$, then Y admits the closed $c-cf$*[2].

2) *If X is locally compact, then the converse of* 1) *is true.*

Proof. We notice that f being open and closed, Y is an M-space [16].

1) The closed $c-cf$ S of X has the form $S=X\cup(\beta X-\mu X)$. Since $(\beta f)^{-1}\mu Y=\mu X$ by Theorem 3.4 [13], we have $(\beta f)^{-1}(\beta Y-\mu Y)=\beta X-\mu X$. Thus $T=Y\cup(\beta Y-\mu Y)$ is a $c-cf$ of Y by Lemma 1.1. The closedness (in T^*) of T^* is obvious.

2) Since Y is an M-space, the closed $c-cf$ T of Y has the form $T=Y\cup(\beta Y-\mu Y)$. On the other hand, X being locally compact and f being open and closed, Y is locally compact and hence we have that $\beta Y-\mu Y$ is compact. Let us put $S=X\cup(\beta X-\mu X)$. From $(\beta f)^{-1}T^*=S^*$, S^* is compact and it is easy to see that S is the closed $c-cf$ of X.

Remark. If X is not locally compact, the converse of 1) is not necessarily true (see (2) of Example 3.1).

§ 3. Examples.

Let Q be the set of all rational numbers, N the set of positive integers and Ω the first uncountable ordinal.

Exampe 3.1. Let f be the projection from $X=Q\times[0,\Omega)$ onto $Y=Q$.

---

2) Using Theorem B, Hoshina pointed out that 1) is true without the assumption "M-ness of X". For, if f is an open quasi-perfect map from X with the closed $c-cf$ S onto Y, then $T=\beta f(S)$ is a $c-cf$ of Y and it is easy to see that T^* is closed in Y^*.

1) f is open and quasi-perfect but not perfect.

2) X is an M-space [4, 7] and $S(X)=X$ is not paracompact and hence X does not admit the closed $c-cf$ [10].

3) Let φ be the identity map from $X(\subset \beta X)$ onto $X(\subset \beta Q\times[0,\Omega])$. Then $S_1=\Phi^{-1}(\beta Q\times[0,\Omega])$ is a $c-cf$ of X by Corollary 1.2 where Φ is the Stone extension of φ. But S_1 is not the closed $c-cf$ of X by 2). Since $Q^*\times[0,\Omega)$ is dense in $\beta Q\times[0,\Omega]$, we have $\mathrm{cl}_{\beta X}S_1^*=\beta X$.

4) Y admits the closed $c-cf$ βY because $Y=\mu Y$ [10, 14].

5) Since f is closed, we have $\mu X=Q\times[0,\Omega]$ [8, 9].

6) $(\beta f)^{-1}Q=\cup\{(\beta f)^{-1}y\,;\,y\in Y\}=\cup\{\mathrm{cl}_{\beta X}f^{-1}(y)\,;\,y\in Y\}=Q\times[0,\Omega]$ $=\mu X$.

7) $S_2=X\cup(\beta f)^{-1}(\beta Y-Y)$ is not a $c-cf$ of X.

Proof. Suppose that S_2 is a $c-cf$ of X. By 5) $\mu X=Q\times[0,\Omega]$. μX is C^*-embedded in βX, and hence $K=\{(\alpha,\Omega)\,;\,\alpha\in Q\}$ is C^*-embedded in βX. Thus $\mathrm{cl}_{\beta X}K$ is homeomorphic with βK and $X^*\cap\mathrm{cl}_{\beta X}K=K^*$ is closed. X being an M-space, μX is of countable type and hence $\beta X-\mu X$ is Lindelöf ([1] or 2.11 of [12]). K^* is a closed subset of $X\cup(\beta X-\mu X)$ which implies the countable compactness of K^*. Thus K^* is cc and Lindelöf, that is, compact which is a contradiction.

Example 3.2. Let X, Y and f be spaces and a map constructed by Morita in [12, [14] and let us put $a_n=(\Omega,\Omega)$ for each $n\in N$. It is proved that 1) f is a perfect map from a locally compact M-space X onto a locally compact space Y which is not an M-space [12] and 2) $\mu Y=Y-\{p\}$ and $\beta Y-\mu Y$ is cc and hence $T=Y\cup(\beta Y-\mu Y)$ is a $c-cf$ of Y [14]. It is obvious that T is not the closed $c-cf$ of Y (For the details of the construction of X, Y, f and the point p, see [12]).

1) $(\beta f)^{-1}(\beta Y-\{p\})=\beta X-\mathrm{cl}_{\beta X}\{a_n\,;\,n\in N\}$ is a $c-cf$ of X by Corollary 1.2 whose X^*-section is not closed in X^*. It is obvious that $X\cup(\beta X-\mu X)$ is the one-point $c-cf$ of X because μX is open in βX ([10, 14]).

2) $\beta f(X\cup(\beta X-\mu X))=\beta Y$ is not a $c-cf$ of Y.

References

[1] M. Henriksen and J. R. Isbell: Some properties of compactifications. Duke Math. J., **25**, 83–105 (1957).

[2] T. Hoshina: On unions of countably-compactifiable spaces. Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A, **12**, 92–94 (1972).

[3] T. Ishii: On closed mappings and M-spaces. I, II. Proc. Japan Acad., **43**, 752–756, 757–761 (1967).

[4] T. Ishii, M. Tsuda, and S. Kunugi: On the products of M-spaces. I, II. ibid., **44**, 897–900, 901–903 (1968).

[5] T. Isiwata: Mappings and spaces. Pacific J. Math., **20**, 455–480 (1967).

[6] ——: Generalization of M-spaces. I, II. Proc. Japan Acad., **45**, 359–363, 364–367 (1969).

[7] T. Isiwata: Products of M-spaces. ibid., **45**, 713–716 (1966).
[8] ——: Topological completions and raelcompactifications. ibid., Supplement to Vol., **47**, 941–947 (1971).
[9] ——: Z-mappings and C^*-embeddings. ibid., **45**, 889–893 (1969).
[10] ——: On closed countably-compactifications. Gen. Topology Appl., **4**, 143–167 (1974).
[11] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).
[12] ——: Some properties of M-spaces. Proc. Japan Acad., **43**, 869–872 (1967).
[13] ——: Topological completions and M-spaces. Sci. Rep. Tokyo Kyoiku Daigaku. Sec. A., **10**, 271–288 (1970).
[14] ——: Countably-compactifiable spaces. ibid., **12**, 7–15 (1972).
[15] ——: Some results on M-spaces (to appear in Proceedings of the conference on topology). Kesthely (1972).
[16] ——: Completion of hyperspaces of compact subsets and Topological Completion of open-closed maps in Gen. Topology Appl., **4**, 217–233 (7974).