6. On Polythetic Groups

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§1. Let G be a locally compact abelian (LCA) group and Z be the additive group of integers. We say G is polythetic if it has a dense subgroup which is a homomorphic image of Z^n. In other words G is to contain n elements $x_1, \cdots, x_n$ such that the subgroup

$$\{m_1x_1+\cdots+m_nx_n\,;\,(m_1,\cdots,m_n)\in Z^n\}$$

is dense in G. We call such elements $x_1, \cdots, x_n$ '*generators of G*'.

In the case $n=1$, G is called *monothetic* and for compact monothetic groups their characterization is stated in terms of their duals by Halmos and Samelson [1]. In this paper we have characterization of LCA polythetic groups by their structures and the smallest numbers of their generators. For the terminologies and notations in this note, see Rudin [2].

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§2. For a LCA polythetic group G let $A(G)$ be the set of integers $n>0$ such that there exists a homomorphic image of Z^n which is dense in G. Clearly $A(G)$ has the smallest element, which we denote by $s(G)$.

Now we state the characterization of compact polythetic groups.

The annihilator Λ of a closed subgroup H of G is the set of all $\gamma\in\Gamma$ (the dual group of G) such that $(x,\gamma)=1$ for all $x\in H$. Λ forms a closed subgroup of Γ.

Lemma 1. *For $i=1,\cdots,n$, let H_i be the closure of the subgroup generated by $x_i\in G$, Λ_i be its annihilator, and let H be the subgroup of G generated by $x_1,\cdots,x_n$. H is dense in G if and only if $\bigcap_{i=1}^{n}\Lambda_i=\{0\}$.*

We denote by T the multiplicative group of all complex numbers of absolute value 1 with the usual topology (or equivalently the additive group of real numbers mod 2π) and by T_d the same group with the discrete topology.

Theorem 1. *Let G be a compact abelian group. G is polythetic if and only if Γ is isomorphic to a subgroup of T_d^n.*

Proof. If G is polythetic, G has generators $x_1,\cdots,x_n$. Since the natural mapping α of T_d^n onto T^n is an algebraic isomorphism, the mapping $\gamma\to\alpha^{-1}((x_1,\gamma),\cdots,(x_n,\gamma))$ is an isomorphism of Γ into T_d^n, because $\alpha^{-1}((x_1,\gamma),\cdots,(x_n,\gamma))=(1,\cdots,1)$ implies $(x_1,\gamma)=\cdots=(x_n,\gamma)=1$ and then $\gamma=0$ (by Lemma 1).

Conversely let Γ be isomorphic to a subgroup of T_d^n, then by expressing α as $\alpha=(\alpha_1,\cdots,\alpha_n)$, where α_i is a continuous homomorphism of T_d^n into T, we can choose $x_i\in G$ $(i=1,\cdots,n)$ such that the restriction of α_i to Γ coinsides with x_i; $\alpha_i(\gamma)=(x_i,\gamma)$ for all $\gamma\in\Gamma$. Since α is one to one, $(x_i,\gamma)=\cdots=(x_n,\gamma)=1$ if and only if $\gamma=0$. Hence by Lemma 1, we have $\bigcap_{i=1}^{n}\Lambda_i=\{0\}$, where Λ_i is the annihilator of the closure of the subgroup generated by x_i. It follows that G is polythetic. Q.E.D.

Now let us recall that the dual group of a compact abelian group is discrete. Theorem 1 is advanced to the following

Theorem 2. *A discrete group Γ is isomorphic to a subgroup of T_d^n if and only if its cardinal number is not greater than the power of the continuum and its torsion group is isomorphic to a subgroup of T_d^n.*

Theorem 3. *Let G be compact and G_0 be the connected component of 0 in G. G is polythetic if and only if G is separable and the totally disconnected factor group G/G_0 is polythetic. And then we have $s(G)=S(G/G_0)$.*

The proofs of these Theorems for $n=1$ are given in [1] and the same proofs hold also for $n\geqq 2$.

§ 3. In this section we consider a non-compact polythetic group G and its polythetic subgroups which are useful to characterize $s(G)$.

We denote by $A\oplus B$ the direct sum of two groups A and B.

Theorem 4. *Every LCA polythetic group G has an open subgroup G' with the following properties;*

i) *G' is the direct sum of a compact group H and R^l $(l\geqq 0)$,*
ii) *$G=G'\oplus Z^k$ for some $k\geqq 0$,*
iii) *$s(G)=s(G')+k$.*

Proof. Put $s(G)=n$ and denote by $\mathfrak{G}$ the dense subgroup of G generated by n elements $x_1,\cdots,x_n\in G$. Since $\mathfrak{G}$ is finitely generated we can assume $x_1,\cdots,x_n$ are independent.

By the principal structure theorem (see [2]), G has an open subgroup G_1 which is the direct sum of a compact group H_1 and R^l $(l\geqq 0)$.

If there exists $x_j\notin G_1$ $(1\leqq j\leqq n)$ such that $m_0x_j\in G_1$ for some m_0 $(\neq 0)\in Z$, then let us consider the subgroup $G_2=\langle G_1, x'_j\rangle$ generated by G_1 and x'_j, where x'_j is an element in G chosen as follows; we express m_0x_j as $m_0x_j=h+r$ $(h\in H_1, r\in R^l)$ and let $x'_j=x_j-r/m_0$.

Since $m_0x'_j\in H_1$, G_2 is a finite union of cosets of G_1, hence is open, and by the same reason the subgroup $H_2=\langle H_1, x'_j\rangle$ is compact. Now, G_2 is the direct sum of H_2 and R^l, because the existence of $m(\neq 0)\in Z$ and $h\in H_1$ such that $h+mx'_j\in R^l$ implies that $m_0(h+mx'_j)\in R^l$ and at the same time that $m_0(h+mx'_j) \in H_1$, which is a contradiction.

After finite steps we obtain an open subgroup $G_k=G'$ which is the direct sum of a compact group $H_k=H$ and R^l such that either $mx_j \in G'$ for all $m \in Z$ or $mx_j \notin G'$ for all $m(\neq 0) \in Z$. It follows $G=G' \oplus Z^k$ for some $k \geqq 0$ and $s(G)=s(G')+k$. Q.E.D.

Theorem 5. *Let G be a LCA polythetic group which is the direct sum of a compact group H and R^l $(l>0)$. Then G has a discrete subgroup D such that*

i) *D is isomorphic to Z^l,*

ii) *the factor group G/D is compact,*

iii) $s(G)=s(G/D)+l$.

Proof. Put $s(G)=n$ and as in the proof of Theorem 4, let $\mathfrak{G}$ be a dense subgroup generated by independent elements x_i $(1 \leqq i \leqq n)$.

x_i is written as $x_i=a_i+b_i$ $(a_i \in H, b_i \in R^l)$ $1 \leqq i \leqq n$.

Since the subgroup $\langle b_1, \cdots, b_n \rangle$ generated by $b_1, \cdots, b_n \in R^l$ is dense in R^l, as vectors in R^l, $\{b_1, \cdots, b_n\}$ contains l linearly independent vectors, which we say $b_1, \cdots, b_l$. Put $D=\langle y_1, \cdots, y_l \rangle$.

Since $\langle b_1, \cdots, b_l \rangle$ is isomorphic to Z^l in R^l, D is discrete and isomorphic to Z^l in G. We show G/D is compact. Let W be a compact neighborhood of 0 in R^l such that $R^l \subset W+\langle b_1, \cdots, b_l \rangle$. Then $G \subset (H \oplus W)+D$. Since $H \oplus W$ is compact we have G/D is compact. Further the structures of $\mathfrak{G}$ and D show that $s(G)=s(D)+s(G/D)$. Q.E.D.

Theorem 5 does not mention any relation between $s(G)$ and $s(H)$. However in the special cases, we can obtain a relation by Theorem 5.

Theorem 6. *If H is compact connected separable and if $G=H \oplus R^l$, then we have $s(G)=s(H) \vee (l+1)$, where $a \vee b=\max\{a, b\}$.*

Proof. The assumptions on H imply $s(H)=1$ (see [1]). Hence $s(H) \vee (l+1)=l+1$ holds for any integer $l>0$. We show $s(G)=l+1$.

By Theorem 5 we can choose a discrete subgroup D such that G/D is compact and $s(G)=s(G/D)+l$. Since G/D is connected and separable we have $s(G/D)=1$. It follows $s(G)=l+1$. Q.E.D.

Theorem 7. *If H is compact separable and is a direct sum of a connected group H_0 and a totally disconnected group H_1, and if $G=H \oplus R^l$, then we have $s(G)=s(H) \vee (l+1)$.*

Proof. We write G as $G=H \oplus G_0$, where $G_0=H_0 \oplus R^l$. Since $s(H) =s(H_0 \oplus H_1)=s(H_1)$ by Theorem 3 and $s(G_0)=l+1$ by Theorem 6, we have

$$s(G) \geqq s(H_1) \vee s(G_0)=s(H) \vee (l+1).$$

Put $m=\min\{s(H_1), s(G_0)\}$ and $n=\max\{s(H_1), s(G_0)\}$. In order to show $s(G) \leqq s(H_1) \vee s(G_0)$, we say that for some $x_1, \cdots, x_n \in G$, $\mathfrak{G}=\langle x_1, \cdots, x_n \rangle$ is dense in G. First we assume $m=s(H_1)$. Let $a_1, \cdots, a_m$ be generators of H_1 and $b_1, \cdots, b_n$ be generators of G_0. Define $x_j \in G$ by

$$x_j = a_j + b_j \qquad \text{if } 1 \leqq j \leqq m$$
$$= b_j \qquad \text{if } m+1 \leqq j \leqq n.$$

Let $\gamma \in \Gamma$ be such that $\gamma = 1$ on $\mathfrak{G}$. γ can be uniquely written as $\gamma = \gamma' + \gamma''$ where $\gamma' \in \Gamma'$ (the dual group of H_1) and $\gamma'' \in \Gamma''$ (the dual of G_0). Hence we have

$$(x_j, \gamma) = (a_j, \gamma')^k (b_j, \gamma'')^k = 1 \qquad \text{for } 1 \leqq j \leqq m \text{ and } k \in Z,$$
$$= (b_j, \gamma'')^k = 1 \qquad \text{for } m+1 \leqq j \leqq n \text{ and } k \in Z.$$

Since γ' has finite order q, putting $k = q$ we have $q\gamma'' \in \bigcap_{i=1}^{n} \Lambda_i$, where Λ_i is the annihilator of the closed subgroup generated by b_i. It follows by Lemma 1 $q\gamma'' = 0$. But since γ'' has infinite order, γ'' must be 0. Hence we have

$$(a_j, \gamma')^k = 1 \qquad \text{for } 1 \leqq j \leqq m \text{ and } k \in Z.$$

Again by Lemma 1, we get $\gamma' = 0$. We have shown that $\gamma = 0$, that is, $\mathfrak{G}$ is dense in G. The proof for $m = s(G_0)$ is obtained similarly.

Q.E.D.

References

[1] P. Halmos and H. Samelson: On monothetic groups. Nat. Acad. Sci. US., **28**, 254–258 (1942).

[2] W. Rudin: Fourier Analysis on Groups. Intersci. Pub. (1960).