5. On the Structure of Infinite Transitive Primitive Lie Algebras

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Introduction. In the present paper, we will study the relation between infinite transitive primitive Lie algebra sheaves and their corresponding global Lie algebras. Our proof is done by using the classification theorem of infinite primitive pseudogroups [3]. In the last section, we will calculate the cohomology groups of certain ideals of the Hamiltonian Lie algebra with coefficients in the adjoint representation.

All the results we get in this paper are the extension of the theorems proved by A. Avez, A. Diaz-Miranda and A. Lichnerowicz [1].

1. Preliminaries. Let M be a connected smooth manifold and $\mathfrak{X}(M)$ the Lie algebra of all global smooth vector fields on M. Our main objects are some Lie subalgebras of $\mathfrak{X}(M)$.

It is well known that there are six classes of infinite transitive primitive pseudogroups in the complex analytic case [3]. Now we describe global smooth Lie algebras corresponding to them.

(1) the Lie algebra of all vector fields (i.e. $\mathfrak{X}(M)$),

(2) the Lie algebra of vector fields of divergence zero,

(3) the Lie algebra of vector fields of constant divergence,

(4) the Lie algebra of vector fields preserving a Hamiltonian structure (the Hamiltonian Lie algebra),

(5) the Lie algebra of vector fields preserving a Hamiltonian structure up to constant factors,

(6) the Lie algebra of vector fields preserving a contact structure (the contact Lie algebra).

Let $L(M)$ be one of the global Lie algebras listed above. We denote by $\mathcal{L}$ the Lie algebra sheaf [5] corresponding to it. Let $\mathcal{L}_p$ be a stalk at $p \in M$. Each $\mathcal{L}_p$ has an infinite dimensional Lie algebra structure. We define the linear mapping φ_p of $L(M)$ to $\mathcal{L}_p$ as follows. For any $X \in L(M)$, $\varphi_p(X)$ denotes a germ of X at p. Then φ_p is clearly a Lie homomorphism. It is known that each φ_p is surjective. See for example [4].

The formal algebra L of $\mathcal{L}$ is the Lie algebra consisting of formal Taylor expansions of vector fields belonging to $\mathcal{L}$. For the precise definition, see [5]. Then we have a canonical Lie homomorphism

$$F_p : \mathcal{L}_p \to L.$$

Since we are in a smooth case, F_p is surjective. We will simply denote by L the formal algebra of $\mathcal{L}$, since $\mathcal{L}_p$ is isomorphic to each other for any point of M. Note that each L is an infinite transitive primitive Lie algebra.

2. Results. Let K be a Lie subalgebra of $\mathfrak{X}(M)$. If K contains no non-trivial abelian ideals, K is said to be semi-simple. This definition is equivalent to that of F.H. Vasilescu [6].

It should be noted that the formal algebras of (1), (2), (4) and (6) are simple Lie algebras, and the formal algebra of (3) (resp. (5)) contains (2) (resp. (4)) as the ideal of codimension one.

Let J be an abelian ideal of $L(M)$. Then using the facts stated above, it can be concluded that $F_p \circ \varphi_p(J)=0$ for any $p \in M$, and we clearly have $J=0$. If J is a finite dimensional ideal of $L(M)$, we can also have $J=0$ by means of the analogous method. Thus we have the following.

Theorem 1. *Let $L(M)$ be one of the global Lie algebras listed in* 1. *Then $L(M)$ is semi-simple. Furthermore, $L(M)$ admits no non-trivial finite dimensional ideals.*

It is well known that a finite dimensional semi-simple Lie algebra is decomposed into the direct sum of its simple ideals. But the following theorem, which is also the extension of the result of [1], tells us that each $L(M)$ can not admit the decomposition stated above.

Theorem 2. *Let M be a connected smooth manifold, and let $L(M)$ be one of semi-simple Lie algebras listed in* 1. *Then any non-trivial ideal I of $L(M)$ can not admit a non-trivial supplemental ideal in $L(M)$.*

Outline of Proof. Suppose that I admits a supplemental ideal J in $L(M)$. Let $Z(I)$ be a centralizer of I in $L(M)$, then we easily have the decomposition: $L(M)=I \oplus J=I \oplus Z(I)$. We define the subset $n(I)$ of M as follows;

$$n(I)=\{p \in M\,;\, F_p \circ \varphi_p(I)=0\}.$$

Under these notations, we have

$$M-n(I)=\{p \in M\,;\, F_p \circ \varphi_p(Z(I))=0\}.$$

Both $n(I)$ and $M-n(I)$ are clearly closed subsets of M. Since M is connected, these facts immediately complete the proof.

3. Cohomology groups of the Hamiltonian Lie algebra and its ideals. Let ω be a symplectic form on M^{2n}, that is, ω is a closed 2-form with $\omega^n \neq 0$. Then the conformal Hamiltonian Lie algebra, which we denote by $L_{csp}(M, \omega)$, consists of all vector fields X satisfying the equation $\mathcal{L}(X)\omega=K_X\omega$, where K_X is a constant depending on X. Let N_1 be a space of smooth functions on M, which have compact supports and satisfy $\int_M f\omega^n = 0$. We denote by L_1 the Lie subalgebra of $L_{csp}(M, \omega)$ whose any element X satisfies $i(X)\omega = -du$ $(u \in N_1)$.

The Lie algebra L_1 is clearly an ideal of $L_{csp}(M, \omega)$. Under these notations above, we have the following.

Theorem 3. *Let $H^1(L_1, L_1)$ be the cohomology group of L_1 with respect to the adjoint representation. Then it is isomorphic to $L_{csp}(M, \omega)/L_1$. If M is compact (in this case, ω is non-exact), $\dim H^1(L_1, L_1)$ is equal to the first Betti number of M, and hence it is finite dimensional. If ω is exact, $H^1(L_1, L_1)$ is infinite dimensional.*

Let $L_{sp}(M, \omega)$ be the Hamiltonian Lie algebra, then we can easily generalize the result of Theorem 3, that is, we can calculate the cohomology group of some ideals of $L_{csp}(M, \omega)$. We will resume it in

Theorem 4. *Let A be an ideal of $L_{csp}(M, \omega)$ such that $L_{sp}(M, \omega) \supset A \supset L_1$. Then $H^1(A, A)$ is isomorphic to $L_{csp}(M, \omega)/A$.*

Remark. It should be noted that $[L_1, L_1] = L_1$. This fact was proved by E. Calabi [2], and it is very useful for the proofs of Theorem 3 and Theorem 4.

References

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