4. *On the Norm Properties on Function Spaces*

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1. Introduction. Let X be a compact Hausdorff space and λ be a regular finite measure on X. Let $\phi_x(t)$ be a continuous strictly increasing function of $t \geqq 0$ for each $x \in X$ with $\phi_x(0)=0$ and $\lim_{t\to\infty} \phi_x(x) = +\infty$. We assume further that for a fixed $t>0$, the function $\phi_x(t)$ of $x \in X$ is always measurable and

$$0 < \inf_{x\in X} \phi_x(t) \leqq \sup_{x\in X} \phi_x(t) < +\infty. \tag{A}$$

We define a so-called N-function: $\Phi_x(u)=\int_0^u \phi_x(t)dt$, $x \in X$. Then, we see easily that $\Phi_x(u)$ is a convex continuous function of $u \geqq 0$ for a fixed x and a measurable function of x for a fixed u. We shall consider the function space $L_{\Phi_x}(X)$ of measurable functions which is a so-called Orlicz-Nakano space. Since $\Phi_x(|f(x)|)$ is a non-negative measurable function of $x \in X$ for all measurable function f (with respect to λ) by assumption, we can define a functional

$$M_{\Phi_x}(f)=\int_X \Phi_x(|f(x)|)d\lambda(x). \tag{B}$$

Let us define a function space of measurable functions

$L_{\Phi_x}(X)=\{f;$ measurable and $M_{\Phi_x}(cf)<+\infty$ for some $c>0\}$.

Now, we shall consider the complementary function $\Psi_x(u)$ for $\Phi_x(u)$ such that

$$\psi_x(t)= \sup_{\phi_x(u)\leqq t} u \qquad (=\phi_x^{-1}(t))$$

and

$$\Psi_x(u)=\int_0^u \psi_x(s)ds \qquad \text{for } x \in X.$$

We see by assumption $\psi_x(t)$ (resp. $\Psi_x(u)$) has the same properties as $\phi_x(t)$ (resp. $\Phi_x(u)$). In our discussion, $\|\cdot\|_{\Phi_x}$ means the norm defined in [3]. In [3], this norm is called the first modular norm.

Corresponding to an equi-measurable transformation in X, L_p and Orlicz spaces are of importance, since the norm of the function in these spaces is invariant under the transformation. But, in many cases which are not expected uniform properties at each point in X and will be occurred in applications, it is natural to consider the spaces $L_{\Phi_x}(X)$, since the property of functions may be changeable under the transformation. H. Nakano considered more wider sense than that of ours.

But we shall consider here the space $L_{\Phi_x}(X)$ more restricted form as above, in order to avoid non-essential discussion. In this note, we shall consider Szegö's type theorem in function space $L_{\Phi_x}(X)$, in order to investigate the norm on $L_{\Phi_x}(X)$.

2. **Main results.** Let E be a subset of the linear space $C_R(X)$ of continuous real-valued functions on X, which satisfies

(i) $c+f\in E$ for any real constant c and $f\in E$,

(ii) E is uniformly dense in $C_R(X)$.

We shall use the letter μ, for a positive regular Borel measure on X such that $\mu(X)=1$.

Lemma 1. *Suppose $f\in L_{\Phi_x}(X)$, $\|f\|_{\Phi_x}\leqq 1$, and the integral $\int_X \log|f(x)|\,d\mu(x)$ is definite even if either finite or infinite. Then, for an arbitrary positive number ε, there exists a function $g\in E$, such that $\|\exp g\|_{\Phi_x}\leqq 1$ and $\int_X g(x)d\mu(x)\geqq\int_X \log|f(x)|\,d\mu(x)-\varepsilon$. $\Big($In the case $\int_X \log|f(x)|\,d\mu(x)=+\infty$, it means that for any positive number α we can find g with $\int_X g(x)d\mu(x)>\alpha$.$\Big)$*

Lemma 2. *If μ is not absolutely continuous with respect to λ, then for any positive number p there exists a function $f\in L_{\Phi_x}$ such that $\|f\|_{\Phi_x}\leqq 1$, and $\int_X \log|f(x)|\,d\mu(x)\geqq p$.*

For each fixed $x\in X$, the function $t\phi_x(t)$, $t\geqq 0$ is a strictly increasing continuous function with the range $[0,+\infty)$, so it has the inverse function defined on $[0,+\infty)$. Denoting it by I_x, it follows that I_x is a continuous increasing function with $I_x^{-1}(t)=t\phi_x(t)$, $s=I_x(s)\phi_x(I_x(s))$, $t, s\geqq 0$. $I_x(f(x))$ is a Borel measurable function on X for any Borel function f on X. Furthermore, let f be a λ-integrable non-negative function such that $0<\int_X f(x)d\lambda(x)$ then the function $F(\alpha)=\int_X \Psi_x(\phi_x(I_x(\alpha f(x)))d\lambda(x)$, $\alpha\geqq 0$ is a strictly increasing continuous function with the range $[0,+\infty)$. Hence it holds the first half of the following lemma, by the intermediate value theorem.

Lemma 3. *Let μ be absolutely continuous with respect to λ. Then there is a unique $\alpha_\mu>0$ such that*

$$\int_X \Psi_x\Big(\phi_x\Big(I_x\Big(\alpha_\mu\frac{d\mu}{d\lambda}(x)\Big)\Big)\Big)d\lambda(x)=1$$

while $\int_X \log I_x\Big(c\frac{d\mu}{d\lambda}(x)\Big)d\mu(x)$ is either finite or equals to $+\infty$, for any $c>0$. Here the notation $\frac{d\mu}{d\lambda}$ means the usual Radon-Nikodym derivative.

Theorem 1. (1) *If μ is absolutely continuous with respect to λ, then there exists a unique $\alpha_\mu > 0$ such that*

$$\inf\left\{\|\exp f\|_{\Phi_x};\, f \in E,\ \int_X f(x)d\mu(x) \geqq 0\right\}$$
$$= \alpha_\mu \exp\left(-\int_X \log I_x\left(\alpha_\mu \frac{d\mu}{d\lambda}(x)\right)d\mu(x)\right).$$

(2) *If μ is not absolutely continuous with respect to λ, then*

$$\inf\left\{\|\exp f\|_{\Phi_x};\, f \in E,\ \int_X f(x)d\mu(x) \geqq 0\right\} = 0.$$

Proof. Firstly, we note that

$$\inf\left\{\|\exp f\|_{\Phi_x};\, f \in E,\ \int_X f(x)d\mu(x) \geqq 0\right\}$$
$$= \exp\left(-\sup\left\{\int_X f(x)d\mu(x);\, f \in E,\ \|\exp f\|_{\Phi_x} \leqq 1\right\}\right)$$
$$= \exp\left(-\sup \int_X \log|f(x)|\,d\mu(x);\, \|f\|_{\Phi_x} \leqq 1\right\}\right)$$

by the definition of E, and by Lemma 1. Next, let h be the function

$$h(x) = \alpha_\mu^{-1} I_x\left(\alpha_\mu \frac{d\mu}{d\lambda}(x)\right), \qquad x \in X$$

where $\alpha_\mu > 0$ is the number determined in Lemma 3. Then, $\|h\|_{\Phi_x} \leqq 1$ and

$$\sup\left\{\int_X \log|f(x)|\,d\mu(x);\, \|f\|_{\Phi_x} \leqq 1\right\} = \int_X \log h\, d\mu.$$

Finally, combining this equality with the preceding equality, it follows

$$\inf\left\{\|\exp f\|_{\Phi_x};\, f \in E,\ \int_X f(x)d\mu(x) \geqq 0\right\}$$
$$= \exp\left(-\sup\left\{\int_X \log|f(x)|\,d\mu(x);\, \|f\|_{\Phi_x} \leqq 1\right\}\right.$$
$$= \alpha_\mu \exp\left(-\int_X \log I_x\left(\alpha_\mu \frac{d\mu}{d\lambda}(x)\right)d\mu(x)\right).$$

It is an obvious matter in the case when μ is not absolutely continuous with respect to λ, by Lemma 2.

3. Logmodular algebra. Let A be a logmodular algebra on X. By A_0 we will denote the kernel $\tau^{-1}(0)$ for a multiplicative linear functional $\tau \neq 0$ on A. Since A is a logmodular algebra, every τ has a unique representing measure m, for which the Jensen's equality holds, i.e. $\log\left|\int_X f(x)dm(x)\right| = \int_X \log|f(x)|\,dm(x)$ for $f \in A^{-1}$.

Theorem 2. (1) *If m is absolutely continuous with respect to λ, then there exists $\alpha_m > 0$ such that*

$$\inf\{\|1+f\|_{\Phi_x};\, f \in A_0\} = \alpha_m \exp\left(-\int_X \log I_x\left(\alpha_m \frac{dm}{d\lambda}(x)\right)dm(x)\right).$$

(2) *If m is not absolutely continuous with respect to λ, then*

$$\inf\{\|1+f\|_{\Phi_x};\, f \in A_0\} = 0.$$

Proof. Set $E=\{\log|f|; f\in A^{-1}\}$ where A^{-1} is the family of invertible elements of A. Then, E is a subset of $C_R(X)$ satisfying (i) and (ii). By the Theorem 1, we obtain

$$I=\inf\left\{\|f\|_{\Phi_x}; f\in A^{-1}, \int_X \log|f(x)|\,dm(x)\geqq 0\right\}$$
$$=\alpha_m \exp\left(-\int_X \log I_x\Big(\alpha_m\frac{dm}{d\lambda}(x)\Big)dm(x)\right)$$

for $\alpha_m>0$. Put $F=\{\log|f|; f\in A\}$. Then F is a subset of the linear space of all Borel measurable functions on X, and satisfies (i) and (ii). Hence, as it is shown, the Lemmas and Theorem 1 for F are valid too. The number α_m depends only on the measure m, hence we obtain

$$J=\inf\left\{\|f\|_{\Phi_x}; f\in A, \int_X \log|f(x)|\,dm(x)\geqq 0\right\}$$
$$=\alpha_m \exp\left(-\int_X \log I_x\Big(\alpha_m\frac{dm}{d\lambda}(x)\Big)dm(x)\right),$$

in particular $J=I$.

Let $A_1=\{1+f; f\in A_0\}$, $A_1^+=\{cf; |c|\geqq 1, f\in A_1\}$.

Since m is a Jensen measure, we obtain

$$\inf\{\|f\|_{\Phi_x}; f\in A_1\}=\inf\{\|f\|_{\Phi_x}; f\in A_1^+\}$$

and

$$J\leqq \inf\{\|f\|_{\Phi_x}; f\in A_1^+\}\leqq I,$$

which give

$$\inf\{\|1+f\|_{\Phi_x}; f\in A_0\}=\alpha_m \exp\left(-\int_X \log I_x\Big(\alpha_m\frac{dm}{d\lambda}(x)\Big)dm(x)\right).$$

If m is not absolutely continuous with respect to λ, Lemma 2 and the proof of the Theorem 1 imply

$$J=I=\exp\left(-\sup\left\{\int_X \log|f(x)|\,dm(x); \|f\|_{\Phi_x}\leqq 1\right\}\right)=0$$

q.e.d.

References

[1] T. Gamelin: Uniform Algebra. Prentice Hall, Englewood Cliffs, N. J. (1969).
[2] K. Hoffman: Banach Spaces of Analytic Functions. Prentice Hall, Englewood Cliffs, N. J. (1962).
[3] H. Nakano: Modulared Semi-Ordered Linear Spaces. Maruzen, Tokyo (1950).
[4] ——: Topology and Linear Topological Spaces. Maruzen, Tokyo (1951).
[5] K. Urbanik: Szegö's theorem for Orlicz spaces. Bull. Acad. Polonaise des Sci., **14**, p. 503–509 (1966).