29. *The Embedding Problem for Operator Groups*

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By a *semigroup* in a Banach space X we mean a one-parameter family $\{T_t : t \geqq 0\}$ of bounded linear operators on X such that (s_1) $T_0 = I$ (the identity operator on X), $T_{t+s} = T_t T_s$ for $t, s \geqq 0$, and (s_2) for $x \in X$, $T_t x$ is strongly measurable for $t > 0$. A one-parameter family $\{G_t : t \in R\}$ of bounded linear operators on X is said to to be a *one-parameter strongly continuous group* in X, if (g_1) $G_0 = I$, $G_{t+s} = G_t G_s$ for $t, s \in R$, and (g_2) for $x \in X$, $G_t x$ is strongly continuous on R with respect to t. Let $\{T_t\}$ be a semigroup in X. We say that the semigroup $\{T_t\}$ can be *embedded* in a group iff there exists a one-parameter strongly continuous group $\{G_t\}$ in X such that $G_t = T_t$ for $t \geqq 0$. A well-known theorem of Hille and Phillips ([1], Theorem 16.3.6.) states that a semigroup $\{T_t\}$ in X can be embedded in a group iff T_{t_0} is injective and surjective for some $t_0 > 0$.*) Our purpose in this paper is to give another version of this theorem in terms of Fredholm operator theory.

Let X and Y be Banach spaces. $B(X, Y)$ will denote the set of all bounded linear operators from X to Y. For basic properties of Fredholm operators, we refer to Schechter [2]. An operator $T \in B(X, Y)$ is said to be *Fredholm* if (f_1) $\alpha(T) \equiv \dim N(T) < \infty$, (f_2) $R(T)$ is closed, and (f_3) $\beta(T) \equiv \dim N(T^*) < \infty$, where $N(T)$, $R(T)$ and T^* denote the null space, the range and the adjoint operator of T, respectively. We denote by $\Phi(X, Y)$ the class of all Fredholm operators from X to Y. For $T \in \Phi(X, Y)$ we define the index $i(T)$ of T by $i(T) = \alpha(T) - \beta(T)$. We shall use the following facts concerning Fredholm operators:

(a) If $T_1 \in \Phi(X, Y)$ and $T_2 \in \Phi(Y, Z)$, then $T_2 T_1 \in \Phi(X, Z)$ and $i(T_2 T_1) = i(T_1) + i(T_2)$.

(b) Assume that $T_1 \in B(X, Y)$ and $T_2 \in B(Y, Z)$ are such that $T_2 T_1 \in \Phi(X, Z)$. If either $\alpha(T_2) < \infty$ or $\beta(T_1) < \infty$, then $T_1 \in \Phi(X, Y)$ and $T_2 \in \Phi(Y, Z)$.

We now state our theorem:

Theorem. *A semigroup* $\{T_t\}$ *in* X *can be embedded in a group iff*

(E_1) $\bigcap_{t>0} N(T_t) = \{0\}$; *and*

*) In [1] the semigroup $\{T_t\}$ is supposed to be of class (A), although it is proved without this assumption that the invertibility of some T_{t_0} implies that of every T_t; hence the theorem holds for every semigroup in X.

(E_2) $T_{t_0} \in \Phi(X, X)$ *for some* $t_0 > 0$.

Proof. If $\{T_t\}$ can be embedded in a group, then each T_t is invertible, so $N(T_t)=\{0\}$ and $T_t \in \Phi(X, X)$ for all $t>0$. Conversely, suppose that (E_1) and (E_2) hold. Then we shall establish the following facts:

(i) $T_t \in \Phi(X, X)$ for all $t>0$;
(ii) $i(T_t)=0$ for all $t>0$; and
(iii) $\alpha(T_t)=0$ for all $t>0$.

Given (i), (ii) and (iii), then $\alpha(T_t)=\beta(T_t)=0$, T_t is invertible for all $t>0$, and by the theorem of Hille and Phillips mentioned above $\{T_t\}$ can be embedded in a group.

Proof of (i). Let $t \in (0, t_0)$. Then $T_t T_{t_0-t} = T_{t_0-t} T_t = T_{t_0}$ and $N(T_t) \subset N(T_{t_0})$. Hence $\alpha(T_t) \leqq \alpha(T_{t_0}) < \infty$, and so $T_t \in \Phi(X, X)$ (by Fact (b)). Let $t > t_0$. Then $t = mt_0 + s$ for some positive integer m and a number $s \in (0, t_0)$. Therefore, $T_t = T_{t_0}^m T_s \in \Phi(X, X)$ (by Fact (a)).

Proof of (ii). For every pair of integers m and n with $0 \leqq m \leqq n$, we have $i(T_{m/n}) = (m/n) i(T_1)$ (by Fact (a)). However, the function $i(T_t)$ is integer-valued, so $i(T_1)=0$ and $i(T_r)=0$ for all rational numbers in $(0, \infty)$. On the other hand, $\alpha(T_t)$ and $\beta(T_t)$ are both nondecreasing as functions of t. For, if $0<s<t$, then $N(T_s) \subset N(T_t)$ and $N(T_s^*) \subset N(T_t^*)$ by the semigroup property (s_1); hence $\alpha(T_s) \leqq \alpha(T_t)$ and $\beta(T_s) \leqq \beta(T_t)$. Since $i(T_t) = \alpha(T_t) - \beta(T_t)$ is integer-valued and of bounded variation on $[0, 1]$, there are at most a finite number of jumps in $[0, 1]$. Let $t_1, t_2, \cdots, t_n$ be the points at which the jumps may occur. Then $i(T_t)$ is constant on the intervals $(0, t_1), (t_1, t_2), \cdots, (t_{n-1}, t_n)$ and $(t_n, 1)$ (where $(0, t_1)$ and $(t_n, 1)$ are empty sets if $t_1=0$ and $t_n=1$). Now $i(T_{t_1}) = 2i(T_{t_1/2}) = 0$ if $t_1>0$, and similarly, $i(T_{t_j})=0$ for $j=2, 3, \cdots, n$. Therefore, $i(T_t) \equiv 0$ on $[0, 1]$, and hence it follows that $i(T_t)=0$ for all $t \geqq 0$.

Proof of (iii). Since $\alpha(T_t)$ is nondecreasing and is integer-valued, there exists a positive number t_1 such that $\alpha(T_t)$ is constant on the interval $(0, t_1)$. Since $N(T_t)$ is nondecreasing, $N(T_t)$ is constant with respect to t in $(0, t_1)$. Hence, $\{0\} = \bigcap_{t>0} N(T_t) = \bigcap_{0<t<t_1} N(T_t) = N(T_s)$ for $s \in (0, t_1)$. This means that T_s is injective and $\alpha(T_s)=0$ for $s \in (0, t_1)$. Let $t>0$. Then there exist a nonnegative integer m and a number $s \in (0, t_1)$ such that $t = mt_1/2 + s$. Since $T_t = T_{t_1/2}^m T_s$, T_t is also injective and $\alpha(T_t)=0$.

Remarks. (1) In our theorem we considered two conditions (E_1) and (E_2). However, condition (E_1) is automatically satisfied for semigroups of basic classes discussed in [1]. Even if (E_1) is not satisfied, we can pass to a semigroup $\{\hat{T}_t\}$ in a quotient space in which condition (E_1) is satisfied. This can be done in the following manner. Let $\{T_t\}$ be a semigroup in X such that $N = \bigcap_{t>0} N(T_t) \neq \{0\}$. Notice that for

$x \in X$, $T_t x$ is strongly continuous on $(0, \infty)$ with respect to t ([1], Theorem 10.2.3). Since N is a closed subspace of X, X/N is a Banach space in a natural way. Let $\nu: X \to X/N$ be a natural mapping and write $[x]$ for the coset containing x. Note that $[x] = \nu x$. Now for $t \geqq 0$, define $\hat{T}_t: X/N \to X/N$ by $\hat{T}_t[x] = \nu T_t x$ for $x \in [x] \in X/N$. Then $\|\hat{T}_t\| \leqq \|T_t\|$ for $t \geqq 0$. Since $\hat{T}_t \nu = \nu T_t$ on X, it is seen that $\{\hat{T}_t : t \geqq 0\}$ forms a semigroup in X/N. We then demonstrate that $\hat{N} = \bigcap_{t>0} N(\hat{T}_t) = \{[0]\}$. If $\hat{T}_t[x] = [0]$ for $t > 0$, then $\nu T_t x = [0]$ for $t > 0$ and $x \in [x]$; hence $T_t x \in N$ for all $t > 0$. This implies that $T_t x = 0$ for all $t > 0$, and so $x \in N$ or $[x] = [0]$. Consequently, condition (E_1) holds for the semigroup $\{\hat{T}_t\}$ in the quotient space X/N. Moreover, if the original semigroup $\{T_t\}$ satisfies (E_2), then so does $\{\hat{T}_t\}$. In fact, let $T_t \in \Phi(X, X)$. Then, $\dim N < \infty$ and $\nu \in \Phi(X, X/N)$. From this it follows that $\hat{T}_t \nu = \nu T_t \in \Phi(X, X/N)$ and $\hat{T}_t \in \Phi(X/N, X/N)$ (by Fact (b)). Thus, given a semigroup $\{T_t\}$ in X satisfying (E_2), we can always associate a one-parameter strongly continuous group with $\{T_t\}$.

(2) If $T \in \Phi(X, X)$, there exist closed subspaces X_0 and Y_0 of X such that $\dim Y_0 = \beta(T)$ and $X = N(T) \oplus X_0 = R(T) \oplus Y_0$; and it is proved ([2], p. 108) that there is an operator T^- with $N(T^-) = Y_0$ and $R(T^-) = X_0$ such that $F_1 = I - T^- T$ and $F_2 = I - T T^-$ are operators of finite rank satisfying $N(F_1) = X_0$, $R(F_1) = N(T)$, $N(F_2) = R(T)$ and $R(F_2) = Y_0$. Hence, T^- is a "pseudo-inverse" of T. (Notice that $T^{-1} \in B(X, X)$ iff $\alpha(T) = \beta(T) = 0$.) The theorem of Hille-Phillips asserts that the semigroup $\{T_t\}$ can be embedded in a group iff T_{t_0} is invertible for some $t_0 > 0$. Our theorem states that $\{T_t\}$ can be embedded in a group iff (E_1) holds and T_{t_0} has a "pseudo-inverse" for some $t_0 > 0$.

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References

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[2] M. Schechter: Principles of Functional Analysis. Academic Press (1971).