80. On the Singularities of the Riemann Functions of Mixed Problems for the Wave Equation in Plane-Stratified Media. II

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In this note we shall continue our study of the Riemann function of the mixed problem (1)–(5) in the previous note [1].

3. **Expression of the secondary Riemann function $F(x, y)$ $(=F_t(x, y)$ in $\Omega_t)$.** The $F_1(x, y)$ and $F_2(x, y)$ are given by the following formulas.

Case $0<y_n<h$.

$$
\begin{aligned}
&(2\pi)^n F_1(x, y)\\
&=\int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle+(x_n-h)\lambda_1^+-y_n\xi_n\}Q(\zeta)\begin{vmatrix} B_1(-\lambda_1^+) & C_1(\lambda_2^+)\\ B_2(-\lambda_1^+) & C_2(\lambda_2^+)\end{vmatrix}\\
&\qquad\qquad\times d\zeta/R(\zeta')P_1(\zeta)\\
&+\int_{S_m} -\exp i\{\langle x'-y', \zeta'\rangle-(x_n-h)\lambda_1^+-y_n\xi_n\}Q(\zeta)\begin{vmatrix} B_1(\lambda_1^+) & C_1(\lambda_2^+)\\ B_2(\lambda_1^+) & C_2(\lambda_2^+)\end{vmatrix}\\
&\qquad\qquad\times d\zeta/R(\zeta')P_1(\zeta)\\
&+\int_{S_m} -\exp i\{\langle x'-y', \zeta'\rangle+x_n\lambda_1^++(h-y_n)\xi_n\}Q(-\lambda_1^+)\begin{vmatrix} B_1(\zeta) & C_1(\lambda_2^+)\\ B_2(\zeta) & C_2(\lambda_2^+)\end{vmatrix}\\
&\qquad\qquad\times d\zeta/R(\zeta')P_1(\zeta)\\
&+\int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle-x_n\lambda_1^++(h-y_n)\xi_n\}Q(\lambda_1^+)\begin{vmatrix} B_1(\zeta) & C_1(\lambda_2^+)\\ B_2(\zeta) & C_2(\lambda_2^+)\end{vmatrix}\\
&\qquad\qquad\times d\zeta/R(\zeta')P_1(\zeta)\\
&\equiv F_{1,1}(x, y)+F_{1,2}(x, y)+F_{1,3}(x, y)+F_{1,4}(x, y), \qquad \text{respectively.}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&(2\pi)^n F_2(x, y)\\
&=\int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle+(x_n-h)\lambda_2^+-y_n\xi_n\}Q(\zeta)\begin{vmatrix} B_1(\lambda_1^+) & B_1(-\lambda_1^+)\\ B_2(\lambda_1^+) & B_2(-\lambda_1^+)\end{vmatrix}\\
&\qquad\qquad\times d\zeta/R(\zeta')P_1(\zeta)\\
&+\int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle+(x_n-h)\lambda_2^++(h-y_n)\xi_n\}\\
&\quad\times\Bigg[\exp\{-ih\lambda_1^+\}Q(\lambda_1^+)\begin{vmatrix} B_1(-\lambda_1^+) & B_1(\zeta)\\ B_2(-\lambda_1^+) & B_2(\zeta)\end{vmatrix}\\
&\quad-\exp\{ih\lambda_1^+\}Q(-\lambda_1^+)\begin{vmatrix} B_1(\lambda_1^+) & B_1(\zeta)\\ B_2(\lambda_1^+) & B_2(\zeta)\end{vmatrix}\Bigg]d\zeta/R(\zeta')P_1(\zeta)\\
&\equiv F_{2,1}(x, y)+F_{2,2}(x, y) \qquad \text{respectively.}
\end{aligned}
\tag{15}
$$

Case $y_n>h$.

$$(2\pi)^n F_1(x, y)$$
$$\begin{aligned}(16)\qquad &= \int_{S_m} -\exp i\{\langle x'-y', \zeta'\rangle + x_n\lambda_1^+ + (h-y_n)\xi_n\} Q(-\lambda_1^+) \begin{vmatrix} C_1(\zeta) & C_1(\lambda_2^+) \\ C_2(\zeta) & C_2(\lambda_2^+) \end{vmatrix} \\ &\qquad\qquad \times d\zeta / R(\zeta')P_2(\zeta) \\ &+ \int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle - x_n\lambda_1^+ + (h-y_n)\xi_n\} Q(\lambda_1^+) \begin{vmatrix} C_1(\zeta) & C_1(\lambda_2^+) \\ C_2(\zeta) & C_2(\lambda_2^+) \end{vmatrix} \\ &\qquad\qquad \times d\zeta / R(\zeta')P_2(\zeta) \\ &\equiv F_{1,1}(x, y) + F_{1,2}(x, y) \qquad \text{respectively.}\end{aligned}$$

$$(2\pi)^n F_2(x, y)$$
$$\begin{aligned}(17)\qquad &= \int_{S_m} \exp i\{\langle x'-y', \zeta'\rangle + (x_n-h)\lambda_2^+ + (h-y_n)\xi_n\} \\ &\times \Bigg[Q(\lambda_1^+) \begin{vmatrix} B_1(-\lambda_1^+) & C_1(\zeta) \\ B_2(-\lambda_1^+) & C_2(\zeta) \end{vmatrix} \exp\{-ih\lambda_1^+\} - Q(-\lambda_1^+) \begin{vmatrix} B_1(\lambda_1^+) & C_1(\zeta) \\ B_2(\lambda_1^+) & C_2(\zeta) \end{vmatrix} \\ &\times \exp\{ih\lambda_1^+\}\Bigg] d\zeta / R(\zeta')P_2(\zeta).\end{aligned}$$

From (11) and (12) we have

$$(18)\qquad 1/R(\zeta') = \sum_{k=0}^{\infty} [R_1(\zeta')]^k \exp\{i(2k+1)h\lambda_1^+\} / [R_0(\zeta')]^{k+1}.$$

Since the case $0<y_n<h$ can be handled in the same way, we consider here only the case $y_n>h$. Substituting (18) in (16) and (17), we find

$$\begin{aligned}(19)\qquad F_{1,1}(x, y) &= \sum_{k=0}^{\infty} \int_{S_m} -\exp i\{\langle x'-y', \zeta'\rangle + (x_n + (2k+1)h)\lambda_1^+ \\ &\quad + (h-y_n)\xi_n\} Q(-\lambda_1^+) \begin{vmatrix} C_1(\zeta) & C_1(\lambda_2^+) \\ C_2(\zeta) & C_2(\lambda_2^+) \end{vmatrix} \\ &\quad \times [R_1(\zeta')]^k d\zeta / [R_0(\zeta')]^{k+1} P_2(\zeta) \equiv \sum_{k=0}^{\infty} F_{1,1,k}(x, y),\end{aligned}$$

respectively, etc.

Every $F_{1,1,k}(x, y)$ has to be interpreted in the sense of distributions with respect to (x, y) (see (14) in [6]). Then it is permissible to deform the integration contour $\zeta_1 = \xi_1 - im \log(2+|\xi'|)$ into the contour $\zeta_1 = \xi_1 - i\gamma$ $(\gamma>0)$. Thus we get

$$\begin{aligned}(20)\qquad F_{1,1,k}(x, y) &= \int_{\mathbf{R}^n} -\exp i\{\langle x'-y', \xi' - i\gamma\vartheta'\rangle + (x_n + (2k+1)h)\lambda_1^+ \\ &\quad + (h-y_n)\xi_n\} Q(-\lambda_1^+) \begin{vmatrix} C_1(\xi - i\gamma\vartheta) & C_1(\lambda_2^+) \\ C_2(\xi - i\gamma\vartheta) & C_2(\lambda_2^+) \end{vmatrix} \\ &\quad [R_1(\xi' - i\gamma\vartheta')]^k d\xi / [R_0(\xi' - i\gamma\vartheta')]^{k+1} P_2(\xi - i\gamma\vartheta).\end{aligned}$$

In the same way we have

$$\begin{aligned}(21)\qquad F_{1,2}(x, y) &= \int_{\mathbf{R}^n} \exp i\{\langle x'-y', \xi' - i\gamma\vartheta'\rangle - (x_n - (2k+1)h)\lambda_1^+ \\ &\quad + (h-y_n)\xi_n\} Q(\lambda_1^+) \begin{vmatrix} C_1(\xi - i\gamma\vartheta) & C_1(\lambda_2^+) \\ C_2(\xi - i\gamma\vartheta) & C_2(\lambda_2^+) \end{vmatrix} \\ &\quad \times [R_1(\xi' - i\gamma\vartheta')]^k d\xi / [R_0(\xi' - i\gamma\vartheta')]^{k+1} P_2(\xi - i\gamma\vartheta) \\ &\equiv \sum_{k=0}^{\infty} F_{1,2,k}(x, y), \qquad \text{respectively.}\end{aligned}$$

$$
\begin{aligned}
(2\pi)^n F_2(x, y) = & \int_{\Xi^n} \exp i\{\langle x'-y', \xi'-i\gamma\vartheta'\rangle + (x_n-h)\lambda_2^+ \\
& + (h-y_n)\xi_n\} Q(\lambda_1^+) \begin{vmatrix} B_1(-\lambda_1^+) & C_1(\xi-i\gamma\vartheta) \\ B_2(-\lambda_1^+) & C_2(\xi-i\gamma\vartheta) \end{vmatrix} \\
& \qquad\qquad \times d\xi / R_0(\xi'-i\gamma\vartheta') P_2(\xi-i\gamma\vartheta) \\
& + \sum_{k=1}^{\infty} \int_{\Xi^n} \exp i\{\langle x'-y', \xi'-i\gamma\vartheta'\rangle + 2kh\lambda_1^+ + (x_n-h)\lambda_2^+ \\
& + (h-y_n)\xi_n\} \Bigg[Q(\lambda_1^+) R_1(\xi'-i\gamma\vartheta') \begin{vmatrix} B_1(-\lambda_1^+) & C_1(\xi-i\gamma\vartheta) \\ B_2(-\lambda_1^+) & C_2(\xi-i\gamma\vartheta) \end{vmatrix} \\
& - Q(-\lambda_1^+) R_0(\xi'-i\gamma\vartheta') \begin{vmatrix} B_1(\lambda_1^+) & C_1(\xi-i\gamma\vartheta) \\ B_2(\lambda_1^+) & C_2(\xi-i\gamma\vartheta) \end{vmatrix} \Bigg] \\
& \qquad \times [R_1(\xi'-i\gamma\vartheta')]^{k-1} d\xi / [R_0(\xi'-i\gamma\vartheta')]^{k+1} P_2(\xi-i\gamma\vartheta) \\
\equiv & \sum_{k=0}^{\infty} F_{2,k}(x, y), \qquad \text{respectively.}
\end{aligned} \tag{22}
$$

Now we can apply the methods or more directly the results in [6], [9]–[11] to the study of the singular supports of $F_{1,1,k}$, $F_{1,2,k}$ and $F_{2,k}$ $(k=0, 1, \cdots)$.

The propagation of singularities (waves) due to a point source $\delta(x''-y'')$ and their reflection and refraction can be visualized by means of figures which represent the location of singularities. Owing to limited space we do not give here the formulas describing explicitly the singular supports of $E(x-y)$ and $F(x, y)$. We only state the followings. In the both cases where $0<y_n<h$ and $y_n>h$, the phenomenon is different according as $a_1>a_2$ or $a_1<a_2$. In the case $y_n>h$, lateral waves appear in general in the medium Ω_{II} when $a_1>a_2$. In the case $0<y_n<h$, lateral waves appear in general in the medium Ω_I when $a_1<a_2$ and they are trapped with reflected waves by reflection at the boundary and at the interface $x_n=h$.

Remark 1. The method in this series of notes can also applied to the study of the Riemann functions of mixed problems in plane-stratified media for more general hyperbolic differential operators treated in [10] and [11], and it is not difficult to obtain the corresponding results.

Remark 2. Our treatment has some analogy with that of a study on the singular supports of fundamental solutions for hyperbolic finite difference-differential operators in Kakita [4]. This was pointed out by Professor Kakita. It seems very interesting to investigate the singularities of fundamental solutions for more general hyperbolic convolution operators (see Kakita [4], [5]).

Reference

[1] Matsumura, M.: On the singularities of the Riemann functions of mixed problems for the wave equation in plane-stratified media. I. Proc. Japan Acad., 52, 289–292 (1976).