77. *On the System of Pfaffian Equations of Briot-Bouquet Type*

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§1. **Introduction.** In this paper we shall extend some well-known results on the system of ordinary differential equations of Briot-Bouquet type to the system of Pfaffian equations. By a system of Pfaffian equations of Briot-Bouquet type we mean a completely integrable system of Pfaffian equations

$$du_i=\sum_{k=1}^{n}\frac{f^{ik}(u_1,\cdots,u_m,x_1,\cdots,x_n)}{x_k}dx_k,\qquad i=1,\cdots,m,$$

or

$$(1)\qquad x_k\frac{\partial u_i}{\partial x_k}=f^{ik}(u,x),\qquad i=1,\cdots,m;\ k=1,\cdots,n,$$

where the f^{ik} are functions holomorphic at the origin $u_1=\cdots=u_m=x_1=\cdots=x_n=0$ and vanishing there. By the use of the usual multi-index notation: $\alpha=(\alpha_1,\cdots,\alpha_m)$, $\beta=(\beta_1,\cdots,\beta_n)$, the Taylor expansions of the f^{ik} are expressible as

$$f^{ik}(u,x)=\sum_{\mu=1}^{m}a_{i\mu}^{k}u_\mu+\sum_{\nu=1}^{n}a_{\nu}^{ik}x_\nu+\sum_{|\alpha|+|\beta|\geq 2}a_{\alpha\beta}^{ik}u^\alpha x^\beta.$$

By denoting A_k the matrix formed by the coefficients of $u_1,\cdots,u_m$ in the developments of $f^{1k},\cdots,f^{mk}$, let $\lambda_1^k,\cdots,\lambda_m^k$ be the eigenvalues of A_k.

The complete integrability condition for (1) can be written as follows:

$$(2)\qquad \sum_{\mu=1}^{m}\frac{\partial f^{il}}{\partial u_\mu}f^{\mu k}+x_k\frac{\partial f^{il}}{\partial x_k}=\sum_{\mu=1}^{m}\frac{\partial f^{ik}}{\partial u_\mu}f^{\mu l}+x_l\frac{\partial f^{ik}}{\partial x_l}.$$

§2. **Formal integration.**

Theorem 2.1. *Suppose that*

(i) *All the A_k, $k=1,\cdots,n$, are similar to diagonal matrices;*

(ii) *For any system of non-negative integers $(\alpha_1,\cdots,\alpha_m,B)$, there exists an index K, $1\leq K\leq n$, such that*

$$\lambda_i^K\neq\sum_{\mu=1}^{m}\alpha_\mu\lambda_\mu^K+B,\qquad i=1,\cdots,m.$$

Then there exists a formal transformation of the form

$$(3)\qquad u_i=\sum_{\mu=1}^{m}p_{i\mu}v_\mu+\sum_{\nu=1}^{n}p_\nu^i x_\nu+\sum_{|\alpha|+|\beta|\geq 2}p_{\alpha\beta}^i v^\alpha x^\beta,$$

which transforms the system (1) into the system

$$(4)\qquad x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k v_i, \qquad i=1,\cdots,m,$$

where $P=(p_{i\mu})\in GL(m,C)$ *and* $\lambda_1^k,\cdots,\lambda_m^k$ *are suitably renumbered for each* k.

Theorem 2.2. *Suppose that there exists an index* K, $1\le K\le n$, *such that*

$$\lambda_i^K\neq\sum_{\mu=1}^m \alpha_\mu\lambda_\mu^K+B, \qquad i=1,\cdots,m,$$

for any system of non-negative integers $(\alpha_1,\cdots,\alpha_m,B)$ *with the exception of the trivial* m *equalities*: $\lambda_i^K=\lambda_i^K$.

Then there exists a formal transformation (3), *which transforms the system* (1) *into the system* (4).

In order to prove Theorems 2.1 and 2.2, it is sufficient to prove the following three lemmata:

Lemma 1. *There exists an invertible linear transformation*

$$u_i=\sum_{\mu=1}^m p_{i\mu}v_\mu, \qquad i=1,\cdots,m,$$

which takes (1) *into a system*

$$x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k v_i+\sum_{\nu=1}^n b_\nu^{ik}x_\nu+\sum_{|\alpha|+|\beta|\ge 2} b_{\alpha\beta}^{ik}v^\alpha x^\beta.$$

Lemma 2. *For a completely integrable system*

$$(5)\qquad x_k\frac{\partial u_i}{\partial x_k}=\lambda_i^k u_i+\sum_{\nu=1}^n a_\nu^{ik}x_\nu+\sum_{|\alpha|+|\beta|\ge 2} a_{\alpha\beta}^{ik}u^\alpha x^\beta,$$

one can find a unique transformation

$$u_i=v_i+\sum_{\nu=1}^n p_\nu^i x_\nu,$$

which transforms (5) *into a system*

$$x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k v_i+\sum_{|\alpha|+|\beta|\ge 2} b_{\alpha\beta}^{ik}v^\alpha x^\beta.$$

Lemma 3. *A completely integrable system of the form*

$$(6)\qquad x_k\frac{\partial u_i}{\partial x_k}=\lambda_i^k u_i+\sum_{|\alpha|+|\beta|\ge N} a_{\alpha\beta}^{ik}u^\alpha x^\beta, \qquad N\ge 2,$$

is transformed by a transformation and only one

$$(7)\qquad u_i=v_i+\sum_{|\alpha|+|\beta|=N} p_{\alpha\beta}^i v^\alpha x^\beta$$

into a system

$$x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k v_i+\sum_{|\alpha|+|\beta|\ge N+1} b_{\alpha\beta}^{ik}v^\alpha x^\beta.$$

Lemma 1 is an immediate consequence of the assumption (i) of Theorem 2.1 or the assumption of Theorem 2.2 and the relations $A_kA_l=A_lA_k$ which are deduced from (2). Lemma 2 is easily proved from the assumption (ii) of Theorem 2.1 or the assumption of Theorem 2.2 and the relations

$$(-\lambda_i^l+\delta_\nu^l)a_\nu^{ik}=(-\lambda_i^k+\delta_\nu^k)a_\nu^{il}$$

which are derived from the complete integrability condition for (5). From the integrability condition for (6) we obtain

$$\left(\sum_{\mu=1}^{m}(\alpha_\mu-\delta_{i\mu})\lambda_i^l+\beta_l\right)a_{\alpha\beta}^{ik}=\left(\sum_{\mu=1}^{m}(\alpha_\mu-\delta_{i\mu})\lambda_i^k+\beta_k\right)a_{\alpha\beta}^{il}. \tag{8}$$

The transformation (7) is invertible as

$$v_i=u_i-\sum_{|\alpha|+|\beta|=N}p_{\alpha\beta}^i u^\alpha x^\beta+\cdots,$$

whence

$$x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k u_i+\sum_{|\alpha|+|\beta|=N}\left(a_{\alpha\beta}^{ik}-\left(\sum_{\mu=1}^{m}\alpha_\mu\lambda_\mu^k+\beta_k\right)p_{\alpha\beta}^i\right)u^\alpha x^\beta+\cdots.$$

Inserting (7) into the right-hand side,

$$x_k\frac{\partial v_i}{\partial x_k}=\lambda_i^k v_i+\sum_{|\alpha|+|\beta|=N}\left(a_{\alpha\beta}^{ik}-\left(\sum_{\mu=1}^{m}(\alpha_\mu-\delta_{i\mu})\lambda_\mu^k+\beta_k\right)p_{\alpha\beta}^i\right)v^\alpha x^\beta+\cdots,$$

from which follows Lemma 3 in virtue of (8).

§ 3. Convergence of formal transformation.

Theorem 3.1. *Suppose that the assumptions* (i), (ii) *of Theorem* 2.1 *and the following assumption are verified*:

(iii) *For each k, $k=1,\cdots,n$, one finds, in the complex plane, a straight line passing through the origin in such a way that the eigenvalues $\lambda_i^k,\cdots,\lambda_m^k$ and unity lie in the same side of the line.*

Then the formal transformation (3) *does converge.*

Theorem 3.2. *The formal transformation* (3) *converges under the assumption of Theorem* 2.2 *and the following*:

(iii)′ *The eigenvalues $\lambda_1^K,\cdots,\lambda_m^K$ and* 1 *lie in the same side of a straight line in the complex plane passing through the origin.*

There is no loss of generality in supposing that the system (1) is of the form

$$x_k\frac{\partial u_i}{\partial x_k}-\lambda_i^k u_i=\sum_{|\alpha|+|\beta|\geq 2}a_{\alpha\beta}^{ik}u^\alpha x^\beta. \tag{9}$$

Then the formal transformation (3) takes the following form:

$$u_i=v_i+\sum_{|\alpha|+|\beta|\geq 2}p_{\alpha\beta}^i v^\alpha x^\beta. \tag{10}$$

Substituting (10) into (9) and using (4), we obtain

$$\left(\sum_{\mu=1}^{m}(\alpha_\mu-\delta_{i\mu})\lambda_\mu^k+\beta_k\right)p_{\alpha\beta}^{ik}=P_{\alpha\beta}(p_{\alpha'\beta'}^i, a_{\alpha''\beta''}^{ik}),$$

where the $P_{\alpha\beta}$ are polynomials in $p_{\alpha'\beta'}^i$, $1\leq i\leq m$, $|\alpha'|+|\beta'|<|\alpha|+|\beta|$, whose coefficients are linear forms in $a_{\alpha''\beta''}^{ik}$, $|\alpha''|+|\beta''|\leq|\alpha|+|\beta|$. We take a convergent power series $\sum_{|\alpha|+|\beta|\geq 2}A_{\alpha\beta}u^\alpha x^\beta$, which is a majorizing series for all $\sum_{|\alpha|+|\beta|\geq 2}a_{\alpha\beta}^{ik}u^\alpha x^\beta$, and set

$$F(u,x)=\sum_{|\alpha|+|\beta|\geq 2}A_{\alpha\beta}u^\alpha x^\beta.$$

Next we choose a positive constant ρ so that we have

$$\left|\sum_{\mu=1}^{m}(\alpha_\mu-\delta_{i\mu})\lambda_\mu^K+\beta_K\right|\geq\rho$$

for some K, $1\leq K\leq n$, and for any (α, β) with $|\alpha|+|\beta|\geq 2$. We see that the system of equations in $u_1, \cdots, u_m$

$$\rho(u_i-v_i)=F(u, x)$$

has a solution expressible by convergent series

$$u_i=v_i+\sum_{|\alpha|+|\beta|\geq 2} P^i_{\alpha\beta}v^\alpha x^\beta$$

and that $\sum_{|\alpha|+|\beta|\geq 2} P^i_{\alpha\beta}v^\alpha x^\beta$ is a majorizing series of $\sum_{|\alpha|+|\beta|\geq 2} p^i_{\alpha\beta}v^\alpha x^\beta$ for $i=1, \cdots, m$.

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Reference

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