75. A Counterexample for the Local Analogy of a Theorem by Iwasawa and Uchida

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Let $\boldsymbol{Q}$ be the rational number field, $\bar{\boldsymbol{Q}}$ the algebraic closure of $\boldsymbol{Q}$ and k_1, k_2 two finite extensions of $\boldsymbol{Q}$ contained in $\bar{\boldsymbol{Q}}$ such that $\mathrm{Gal}(\bar{\boldsymbol{Q}}/k_1)\cong \mathrm{Gal}(\bar{\boldsymbol{Q}}/k_2)$ as topological groups. As K. Iwasawa and K. Uchida have independently proved (cf. [2], [6]), k_1 and k_2 are conjugate over $\boldsymbol{Q}$. In this paper we prove that the analogy for local number fields is not valid: Let p be a prime number, $\boldsymbol{Q}_p$ the field of p-adic numbers, $\bar{\boldsymbol{Q}}_p$ the algebraic closure of $\boldsymbol{Q}_p$. Then there are finite extensions K_1, K_2 of $\boldsymbol{Q}_p$ contained in $\bar{\boldsymbol{Q}}_p$ such that $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_1)\cong \mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_2)$ as topological groups, and that K_1 and K_2 are not conjugate over $\boldsymbol{Q}_p$.

§1. **Preliminaries.** Let K be a local field, i.e. a commutative field which is complete with respect to a discrete valuation, and L/K a finite Galois extension with $G=\mathrm{Gal}(L/K)$ such that the extension of their residue class fields is separable. Let v_L be the normalized discrete valuation of L, and put $A_L=\{a\in L \mid v_L(a)\geqq 0\}$ and $G_x=\{s\in G \mid v_L(s(a)-a)\geqq x+1 \text{ for all } a\in A_L\}$ for $x\geqq -1$. The function $\varphi_{L/K}(t)$ for $t\geqq -1$ is given by

$$\varphi_{L/K}(t)=\int_0^t \frac{dx}{(G_0 : G_x)}.$$

Let $\psi_{L/K}$ be the inverse function of $\varphi_{L/K}$ and put $G^x=G_{\psi_{L/K}(x)}$. A real number $x\geqq -1$ is called a ramification number of L/K (an upper ramification number of L/K, respectively) if $(G_x : \bigcup_{\varepsilon>0} G_{x+\varepsilon})>1$ (if $(G^x : \bigcup_{\varepsilon>0} G^{x+\varepsilon})>1$, respectively). When L/K has only one ramification number x, x is also the only one upper ramification number of L/K and vice versa. L/K is totally ramified if and only if $G=G_0$.

Lemma (cf. [4], p. 197 and p. 198). *Let K_i/K be a cyclic extension of degree p with only one upper ramification number t_i for $i=1,2$, where p is the characteristic of the residue class field of K. Assume that the residue class field extension of K_i/K is separable. Put $M=K_1K_2$. If $t_1\neq t_2$, M/K_2 is a cyclic extension of degree p with only one upper ramification number $\psi_{K_2/K}(t_1)$.*

In the above situation, we remark that M/K_2 is totally ramified if K_1/K is totally ramified.

Let ζ_n be a primitive n-th root of 1 in $\bar{\boldsymbol{Q}}_p$.

Let $K/\boldsymbol{Q}_p$ be a finite extension contained in $\bar{\boldsymbol{Q}}_p$ of degree m with residue degree f. Let K_{tr} be the maximal tamely ramified extension of K in $\bar{\boldsymbol{Q}}_p$. The Galois group of K_{tr}/K is the total completion of a group generated by two elements s, t satisfying a unique relation $s^{-1}ts = t^{p^f}$ (cf. [1], p. 463). Let r be the largest integer such that a primitive p^r-th root of 1 is contained in K_{tr}. Put $q=p^r$. Let c be an integer such that $\zeta_q^c=\zeta_q^s$, and τ the uniquely determined p-adic integer such that $\tau^{p-1}=1$ and $\zeta_q^\tau=\zeta_q^t$. We fix c for each K. Then the Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K)$ is determined by m, f, q, c, τ for odd p and by m, f, q, c (if $q\geqq 4$) for $p=2$ as a topological group (cf. [3], [7]). If $K(\zeta_q)=K$, we assume $c=1$ in § 2.

§ 2. Examples. Theorem. (i) *Let p be odd. Denote by K_0*

$$\boldsymbol{Q}_p(\sqrt[p]{\zeta_p-1}),$$

and by K_i for $i=1, 2, \cdots, p-2$

$$\boldsymbol{Q}_p(\zeta_p, \alpha_i) \text{ with } \alpha_i \in \bar{\boldsymbol{Q}}_p \text{ such that } \alpha_i^p-\alpha_i=\left(\frac{1}{\zeta_p-1}\right)^i.$$

Then $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_i)$ and $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_j)$ are topologically isomorphic for $0\leqq i, j\leqq p-2$, but K_i and K_j are not conjugate over $\boldsymbol{Q}_p$ if $i\neq j$.

(ii) *Denote by L_0*

$$\boldsymbol{Q}_2(\sqrt{\zeta_4-1}),$$

and by L_1

$$\boldsymbol{Q}_2(\beta) \text{ with } \beta \in \bar{\boldsymbol{Q}}_2 \text{ such that } \beta^2-\beta=\frac{1}{\zeta_4-1}.$$

Then $\mathrm{Gal}(\bar{\boldsymbol{Q}}_2/L_0)$ and $\mathrm{Gal}(\bar{\boldsymbol{Q}}_2/L_1)$ are topologically isomorphic, but L_0 and L_1 are not conjugate over $\boldsymbol{Q}_2$.

Proof. (i) By [5], p. 79 and p. 80, $K_i/\boldsymbol{Q}_p(\zeta_p)$ is a totally ramified cyclic extension of degree p with only one upper ramification number p for $i=0$ and i for $i=1, 2, \cdots, p-2$. By [5], p. 86, $\boldsymbol{Q}_p(\zeta_{p^2})/\boldsymbol{Q}_p(\zeta_p)$ has the only one upper ramification number $p-1$. Let $m_i, f_i, q_i, c_i, \tau_i$ for K_i be as in § 1. We have $m_i=p(p-1), f_i=1$. Since by Lemma $K_i(\zeta_{p^2})/K_i$ is totally ramified of degree p, we have $q_i=p$ and so $c_i=\tau_i=1$. Therefore $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_i)$ and $\mathrm{Gal}(\bar{\boldsymbol{Q}}_p/K_j)$ are topologically isomorphic for $0\leqq i, j\leqq p-2$ (cf. § 1). On the other hand, for $s \in \mathrm{Gal}(\bar{\boldsymbol{Q}}_p/\boldsymbol{Q}_p)$ we have $s(K_0)=\boldsymbol{Q}_p(\sqrt[p]{s(\zeta_p)-1})$ and $s(K_i)=\boldsymbol{Q}_p(s(\zeta_p), s(\alpha_i))$ with $s(\alpha_i) \in \bar{\boldsymbol{Q}}_p$ such that $(s(\alpha_i))^p-s(\alpha_i)=\left(\frac{1}{s(\zeta_p)-1}\right)^i$ for $i=1, 2, \cdots, p-2$. Counting the upper ramification number (cf. [5], p. 79 and p. 80), we have K_i and K_j are not conjugate over $\boldsymbol{Q}_p$ if $i\neq j$. (ii) Similarly $L_i/\boldsymbol{Q}_2(\zeta_4)$ is a totally ramified quadratic extension with only one upper ramification number 4 if $i=0$ and 1 if $i=1$. By [5], p. 86, $\boldsymbol{Q}_2(\zeta_8)/\boldsymbol{Q}_2(\zeta_4)$ has the only one upper ramification number 3. Let m_i, f_i, q_i, c_i for L_i be as in § 1. By the same way as in (i), counting $m_i=4, f_i=1, q_i=4, c_i=1$ we have that

Gal $(\bar{\boldsymbol{Q}}_2/L_0)$ and Gal $(\bar{\boldsymbol{Q}}_2/L_1)$ are topologically isomorphic, and that L_0 and L_1 are not conjugate over $\boldsymbol{Q}_2$.

Remark. Let K_i be a local field of characteristic $p>0$ with finite residue class field k_i for $i=1,2$. Let K_i^{ab} be the maximal abelian extension of K_i for $i=1,2$. Assume that Gal (K_1^{ab}/K_1) and Gal (K_2^{ab}/K_2) are topologically isomorphic. Then K_1 and K_2 are isomorphic.

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