93. *Finiteness Theorem for Holonomic Systems of Micro-differential Equations*

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It is known that the solution space of a holonomic system (=maximally overdetermined system) of linear *differential* equations enjoys a nice finiteness property (Kashiwara [2]). This result naturally raises an interesting question whether analogous results hold for holonomic systems of micro-differential equations (=pseudo-differential equations.) Of course, we should talk about the microfunction solutions in this case and this makes the situations complicated.

However, we can overcome the difficulties by making use of a recent result on the boundary value problem for elliptic systems (Kashiwara-Kawai [4]) on one hand and the concrete representation of the action of micro-differential operators on microfunctions (Kashiwara-Kawai [3] and Bony-Schapira [1]) on the other hand.

Our result is the following

Theorem. *Let M be a real analytic manifold, $\mathcal{C}$ the sheaf of microfunctions and $\mathcal{E}$ the sheaf of micro-differential operators. Let $\mathcal{M}$ be a holonomic system of micro-differential equations defined in a neighborhood of a point p of the pure imaginary cotangent bundle $\sqrt{-1}T^*M$. Then, the dimension of the vector space $\mathcal{E}xt^j_{\mathcal{E}}(\mathcal{M}, \mathcal{C})_p$ is finite for any j.*

We can prove this theorem in the following manner.

(I) Define a real hypersurface S in $\boldsymbol{C}^{n+1}$ by $\{(t, z) \in \boldsymbol{C}^{n+1}; \operatorname{Re} t = |z|^2\}$. Set $\Omega = \{(t, z) \in \boldsymbol{C}^{n+1}; \operatorname{Re} t > |z|^2\}$. We define $\mathcal{C}'$ by the inductive limit of $\mathcal{O}(U \cap \Omega)/\mathcal{O}(U)$, where U runs over a fundamental system of neighborhoods of $(t, z) = (0, 0)$. Then we can find an isomorphism between $\mathcal{E}_{M,p}$ and $\mathcal{E}_{\boldsymbol{C}^{n+1},(0,0;-dt)}$ and an isomorphism between $\mathcal{C}_{M,p}$ and $\mathcal{C}'$ so that the action of $\mathcal{E}_{M,p}$ on $\mathcal{C}_{M,p}$ is compatible with that of $\mathcal{E}_{\boldsymbol{C}^{n+1},(0,0;-dt)}$ on $\mathcal{C}'$. (Kashiwara-Kawai [3] §2.1.)

Further, we can choose these isomorphisms so that the characteristic variety Λ of the $\mathcal{E}_{\boldsymbol{C}^{n+1},(0,0;-dt)}$-module $\mathcal{M}'$ corresponding to $\mathcal{M}$ is finite over $\boldsymbol{C}^{n+1}$, since the characteristic variety of $\mathcal{M}$ is Lagrangian.

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(II) Let δ be a sufficiently small fixed positive number. Let R_δ be the set of all micro-differential operators $P(t, z, D_t, D_z)=\sum_{j\geqq 1} a_j(t, z, D_z)D_t^{-j}$ of order $\leqq -1$ satisfying the following property:

$P(t, z, D_t, D_z)$ is a polynomial in D_z and $A_P(t, t', z, D_z)=\sum_{j\geqq 1}\frac{1}{(j-1)!}\times(t-t')^{j-1}a_j(t, z, D_z)$ is a differential operator defined for $|t|, |t'|, |z|<\delta$.

Let c be a positive number such that $c<\delta$. Let U be an open set in $\{(t, z)\in C^{n+1}; |t|, |z|<\delta\}$ which satisfies the following properties:

(1) $U(z_0)=\{t\in C; (t, z_0)\in U\}$ is convex.

(2) $U(z_0)$ is void if $U(z_0)$ does not contain c.

Then any $P(t, z, D_t, D_z)$ in R_δ acts on $\mathcal{O}(U)$ by the following rule:

$$P(t, z, D_t, D_z)f(t, z)=\int_c^t A_P(t, t', z, D_z)f(t', z)dt'. \tag{3}$$

(See Kashiwara-Kawai [3] and Bony-Schapira [1].) Note that, if P and Q belong to R_δ, then PQ belongs to R_δ and that $A_{PQ}(t, t', z, D_z)=\int_{t'}^t A_P(t, t'', z, D_z)A_Q(t'', t', z, D_z)dt''$ holds. Hence $P(Qf)=(PQ)f$ holds. (Bony-Schapira [1])

(III) Since the characteristic variety Λ of $\mathcal{M}'$ satisfies $\Lambda\cap\pi^{-1}(0)=Cdt$ for $\pi: T^*C^{n+1}\to C^{n+1}$, we can find a resolution of $\mathcal{M}'$ in the following form:

$$0\longleftarrow\mathcal{M}'\longleftarrow\mathcal{E}_{C^{n+1}}^{N_0}\xleftarrow{P_0}\mathcal{E}_{C^{n+1}}^{N_1}\xleftarrow{P_1}\mathcal{E}_{C^{n+1}}^{N_2}\longleftarrow\cdots,$$

where P_j's are micro-differential operators contained in R_δ for a sufficiently small $\delta>0$.

(IV) Define Ω_ρ by $\{(t, z)\in C^{n+1}; \operatorname{Re} t>|z|^2+\rho\}$ and U_ε by $\{(t, z)\in C^{n+1}; \operatorname{Re} t>a|z|^2+|\operatorname{Im} t|^2-\varepsilon\}$ for $\rho, \varepsilon>0$. Here a is a fixed constant >1. Then for some $\alpha>0$, $\partial(\Omega_\rho\cup U_\varepsilon)$ $(0<\rho<\varepsilon^\alpha, \varepsilon\ll 1)$ is non-characteristic with respect to $\mathcal{M}'$ in a fixed neighborhood of the origin, that is, the conormal set of $\partial(\Omega_\rho\cup U_\varepsilon)$ is disjoint from Λ. We take a convex open set $\tilde{U}_\varepsilon$ satisfying the conditions (1) and (2) and $\tilde{U}_\varepsilon=U_\varepsilon$ for $\operatorname{Re} t<c$. Then $\tilde{U}_\varepsilon\cap\Omega_\rho$ also satisfies conditions (1) and (2). Therefore, any element in R_δ operates on $\mathcal{O}(\tilde{U}_\varepsilon)$ and $\mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)$, and hence we get complexes:

$$\begin{array}{ccccccccc}
\mathcal{O}(\tilde{U}_\varepsilon)^\cdot & : & \mathcal{O}(\tilde{U}_\varepsilon)^{N_0} & \xrightarrow{P_0} & \mathcal{O}(\tilde{U}_\varepsilon)^{N_1} & \xrightarrow{P_1} & \mathcal{O}(\tilde{U}_\varepsilon)^{N_2} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^\cdot & : & \mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^{N_0} & \xrightarrow{P_0} & \mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^{N_1} & \xrightarrow{P_1} & \mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^{N_2} & \longrightarrow & \cdots
\end{array}$$

Then, we have

$$\varinjlim_{\varepsilon}\varprojlim_{\rho} H^j(\mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^\cdot/\mathcal{O}(\tilde{U}_\varepsilon)^\cdot)=\mathcal{E}xt^j_{\mathcal{E}_{C^{n+1}}}(\mathcal{M}', C')=\mathcal{E}xt^j_{\mathcal{E}_M}(\mathcal{M}, C)_p$$

(V) Making use of the assertion of the non-characteristicness given in (IV), we conclude that

$$H^j(\mathcal{O}(\tilde{U}_\varepsilon\cap\Omega_\rho)^\cdot/\mathcal{O}(\tilde{U}_\varepsilon)^\cdot)\to H^j(\mathcal{O}(\tilde{U}_{\varepsilon'}\cap\Omega_{\rho'})^\cdot/\mathcal{O}(\tilde{U}_{\varepsilon'})^\cdot)$$

is an isomorphism when $0<\varepsilon'<\varepsilon\ll 1$, $\rho'<\rho$, $0<\rho<\varepsilon^\alpha$ and $0<\rho'<\varepsilon'^\alpha$.

Therefore, we have

$$H^j(\mathcal{O}(\overline{\tilde{U}_\varepsilon \cap \Omega_\rho})^\cdot / \mathcal{O}(\bar{\tilde{U}}_\varepsilon)^\cdot) \xrightarrow{\sim} H^j(\mathcal{O}(\tilde{U}_\varepsilon \cap \Omega_\rho)^\cdot / \mathcal{O}(\tilde{U}_\varepsilon)^\cdot)$$

Then, by making use of functional analysis (see Kawai [5], for example), we can conclude that $H^j(\mathcal{O}(\tilde{U}_\varepsilon \cap \Omega_\rho)^\cdot / \mathcal{O}(\tilde{U}_\varepsilon)^\cdot)$ is of finite dimension. Furthermore, since we have

$$\mathcal{E}xt^j_{\mathcal{E}_M}(\mathcal{M}, C)_p = H^j(\mathcal{O}(\tilde{U}_\varepsilon \cap \Omega_\rho)^\cdot / \mathcal{O}(\tilde{U}_\varepsilon)^\cdot)$$

for any $0 < \rho < \varepsilon^\alpha$, $0 < \varepsilon \ll 1$, we get the required result.

The detailed argument of this note will appear somewhere else.

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