90. *Paley-Wiener Type Theorem for the Heisenberg Groups*

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1. The simply connected Heisenberg group G of n-th order consists of elements $g(x, y, z)$ $(x, y \in R^n, z \in R)$ with multiplication law $g(a, b, c) \cdot g(x, y, z) = g(x+a, y+b, z+c+\langle a, y\rangle)$, where $\langle a, y\rangle = \sum_{i=1}^{n} a_i y_i$.

In this paper we state a Paley-Wiener type theorem for the group G by the same method as in [3]. Let N and A be the subgroups of elements $n = g(0, b, c)$ and $a = g(a, 0, 0)$, respectively. Then $G = N \cdot A$ is a semidirect product. On the set $\hat{N}$ of not necessarily unitary characters of N co-adjoint action of A is defined by $a^* \cdot \chi(n) = \chi(ana^{-1})$, $(a \in A, \chi \in N)$. Every irreducible unitary representation of infinite dimension is realized up to equivalence in $L^2(R^n, dx)$ cf. [1], [2]: for $\lambda \neq 0$,

(1) $$T_g^\chi \varphi(g) = e^{\langle \mu, b\rangle} e^{\lambda(\langle b, x\rangle + c)} \varphi(x+a), \text{ for } g = g(a, b, c),$$

which is induced from a unitary character $\chi = (\mu, \lambda)$ of N such that $\chi(g(0, b, c)) = \exp(\langle \mu, b\rangle + \lambda c)$, $(\mu \in \sqrt{-1} \cdot R^n, \lambda \in \sqrt{-1} \cdot R)$. Let $\mathcal{C}$ be the space of functions φ on R^n with finite seminorms $\|\cdot\|_t$ for any $t \in R^n$, where

$$\|\varphi\|_t = \left(\int_{R^n} \exp\langle t, |x|\rangle \cdot |\varphi(x)|^2 dx\right)^{1/2}, \qquad (|x| = (|x_i|)_i).$$

In the space $(\mathcal{C}, \|\cdot\|_t)$ the formula (1) gives a representation $\mathcal{D}_\chi$. Especially we have $\|T_g^\chi \cdot \varphi\|_t \leqq C^\chi(t, g) \|\varphi\|_{\tau^\chi(t, g)}$, $(\varphi \in \mathcal{C})$, with constants $C^\chi(t, g)$ and $\tau^\chi(t, g)$ independent of φ. From easy argument of the existence of invariant bilinear forms follows

Proposition. (i) *A continuous linear operator commuting with all T_g^χ $(g \in G)$ is a scalar multiple of the identity.* (ii) *Representation $\mathcal{D}_\chi$ extends to a unitary one if and only if so is χ* (cf. [4]).

2. Let $Q_{\alpha,\beta,\gamma}$ be a compact set in G of the form

$$\{g(x, y, z)\,;\, |x_i| \leqq \alpha_i, |y_j| \leqq \beta_j, |z| \leqq \gamma, i, j = 1 \cdots n\}.$$

We assign auxiliary functions to $Q = Q_{\alpha,\beta,\gamma}$, $\tau^\chi(t\,; Q) = t + 2\beta |Re\lambda|$, and $C^\chi(t, Q) = \exp[\langle \beta, |Re\mu|\rangle + \gamma |Re\lambda| + 2^{-1}\langle |\tau^\chi(t\,; Q)|, \alpha\rangle]$.

Lemma. *If the support of a function $f \in L^1(G)$ is contained in the compact set Q, the Fourier transform of f: $T_f^\chi = \int_G f(g) T_g^\chi dg$, converges strongly in $\mathcal{C}$ for every $\chi \in \hat{N}$ and it holds*

(2) $$\|T_f^\chi \varphi\|_t \leqq C^\chi(t\,; Q) \|f\|_{L^1} \|\varphi\|_{\tau^\chi(t; Q)} \qquad (t \in R^n).$$

The Plancherel formula takes the following form:

$$\|f\|_{L^2}^2=(2\pi)^{-2}\int_R \|T_f^{(0,\sqrt{-1}\lambda)}\|^2{}_{H-S}\lambda d\lambda,$$

$(\|\cdot\|_{H-S}$; Hilbert-Schmidt norm).

Let $C_0^\infty(Q)$ be the space of C^∞-functions with support contained in Q and $\mathscr{B}(Q)$ be the set of operator-valued functions T^χ of χ having the properties (1°) (2°) and (3°) mentioned below. $\mathscr{B}(Q)$ becomes a Fréchet space with the canonical topology cf. [3].

(1°) *For any polynomials P_k $(k=1,2)$ of $\partial/\partial x_i$ and $(\mu+\lambda x)_j$ there exists a constant $C(P_1,P_2)$ independent of t,χ and φ such that*

$\|P_1\cdot T^\chi\cdot P_2\cdot\varphi\|_t\leqq C(P_1,P_2)C^\chi(t;Q)\|\varphi\|_{\tau^\chi(t;Q)},$ *for every $t\in R^n$.*

(2°) Equivalence relation: $T_a\cdot T^\chi\cdot T_a^{-1}=T^{a^*\cdot\chi}$, ($a\in A, T_a$; *translation by a*).

Now let $\mathcal{C}_\infty$ be the space of $\psi\in\mathcal{C}$ whose distribution derivatives $P(\partial/\partial x)\cdot\psi$ all belong to $\mathcal{C}$ for any polynomial P. We take as its system of seminorms $\|\psi\|_{t,P}=\|P(\partial/\partial x)\psi\|_t$.

(3°) Weak analyticity: *for any $\varphi\in\mathcal{C}$ and $F\in\mathcal{C}'_\infty$; the dual space of $\mathcal{C}_\infty$, $\langle F, P_1\cdot T^\chi P_2\varphi\rangle$ is an entire function of $\chi\in\hat{N}\cong C^n\times C$, for such P_1 and P_2 as in* (1°).

Paley-Wiener type theorem. *Fourier transformation $f\mapsto T_f^\chi$ is a topological isomorphism of $C_0^\infty(Q_{\alpha,\beta,\gamma})$ onto $\mathscr{B}(Q_{\alpha,\beta,\gamma})$.*

3. We sketch only the proof of surjectivity. As showed in [3], the operator T^χ has an integral kernel $H^\chi(x,a)$: $T^\chi\varphi(x)=\int_{R^n}H^\chi(x,a)\varphi(a)\,da$ $(\varphi\in\mathcal{C})$. $H^\chi(x,a)$ is a C^∞-function of a for fixed x and χ and its support is contained in $\{a;\max.(|x_i|-|\alpha_i|,0)\leqq|a_i|\leqq|x_i|+\alpha_i, i=1\cdots n\}$. Let $\chi_{[-\alpha,a_0]}$ be the indicator of a set $\{x; -\alpha_i\leqq x_i\leqq(a_0)_i, i=1\cdots n\}$. We can show for differential operator D_a, $D_a\cdot H^\chi(0,a_0)=(T^\chi\cdot D_a^*\cdot\chi_{[-\alpha,a_0]})(0)$, where D_a^* is the adjoint operator of $\left(\prod_{i=1}^n\partial/\partial a_i\right)\cdot D_a$. By the property (1°) we can extend $T^\chi\cdot D_a^*$ defined on $\mathcal{C}_\infty$ to the whole space $\mathcal{C}$. Put $\psi=T^\chi\cdot D_a^*\cdot\chi_{[-\alpha,a_0]}$, then $\psi\in\mathcal{C}_\infty$ and its support lies in $\{x;|x_i|\leqq 2\alpha_i, i=1\cdots n\}$. We estimate $\psi(0)$ as follows. First for every $s=(s_j)\in R^n$

$$\psi(0)=\int_{-2\alpha_1}^0\cdots\int_{-2\alpha_n}^0\left(\prod_{i=1}^n\partial/\partial x_i\right)(e^{\langle s,x\rangle}\psi(x))\,dx$$
$$=\int_{-2\alpha_1}^0\cdots\int_{-2\alpha_n}^0 e^{\langle s,x\rangle}\prod_{i=1}^n(\partial/\partial x_i+s_i)\cdot\psi(x)\,dx.$$

Schwarz' inequality and application (1°) to $\prod(\partial/\partial x_i+s_i)\cdot\psi$ brings us

$$|\psi(0)|\leqq C(s)\cdot C^\chi(t;Q)\cdot\|\chi_{[-\alpha,a_0]}\|_{\tau^\chi(t;Q)}\left(\int_{-2\alpha_1}^0\cdots\int_{-2\alpha_n}^0 e^{\langle 2s-t,x\rangle}dx\right)^{1/2}.$$

Since $C(s)$ is a polynomial, we have for every $\varepsilon>0$

$|\psi(0)|\leqq C_\varepsilon\cdot\exp[\langle\beta,|Re\mu|\rangle+(\gamma+\langle\varepsilon,\beta\rangle)|Re\lambda|]$, where C_ε is independent of χ. Moreover, choosing suitable polynomials P_1 and P_2 we can also show

$$|\mu^p\lambda^q D_a^r\cdot H^\chi(0,a)|\leqq C_{p,q,r,\varepsilon}\exp[\langle\beta,|Re\mu|\rangle+(\gamma+\langle\varepsilon,\beta\rangle)|Re\lambda|],$$

for any p_i, q_j, r_k and $\varepsilon>0$ and constant $C_{p,q,r,\varepsilon}$ is independent of χ. This shows the following function is the desired one:

$$f(g)=f(a, b, c)=(2\pi)^{-n-1}\int_{R^n\times R} H^{(i\mu, i\lambda)}(0, a)e^{-i[\langle\mu, b\rangle+\lambda c]}d\lambda d\mu,$$

$(i=\sqrt{-1})$. Once surjectivity is proved, topological isomorphism easily follows from Banach-Steinhauss theorem, cf. [3].

Added to Proposition. (iii) *Suppose L is a continuous linear operator in C and characters χ_1 and χ_2 are both in general position. If it holds for all $g\in G$, $L\cdot T_g^{\chi_1}=T_g^{\chi_2}\cdot L$, then $L=0$ or $\chi_2=a^*\chi_1$ for some $a\in A$ and $L=$ const. T_a.*

References

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