# 114. On Holomorphically induced Representations of Exponential Groups

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1976)

The aim of this note is to generalize to the case of exponential groups the results announced in [2] on holomorphically induced representations of split solvable Lie groups.

**1.** Let $G=\exp \mathfrak{g}$ be an exponential group (for the definition, see [6] for example) with Lie algebra $\mathfrak{g}$, $f$ a linear form on $\mathfrak{g}$, $\mathfrak{h}$ a positive polarization of $\mathfrak{g}$ at $f$, $\rho(f, \mathfrak{h})$ the holomorphically induced representation of $G$ constructed from $\mathfrak{h}$ and let $\mathcal{H}(f, \mathfrak{h})$ be the space of $\rho(f, \mathfrak{h})$ [1].

In this note, we find a necessary and sufficient condition on $(f, \mathfrak{h})$ for the non-vanishing of $\mathcal{H}(f, \mathfrak{h})$. We then show that $\rho(f, \mathfrak{h})$ $(\neq 0)$ is irreducible if and only if the Pukanszky condition is satisfied, and that in this case $\rho(f, \mathfrak{h})$ is independent of $\mathfrak{h}$. For reducible $\rho(f, \mathfrak{h})$, we describe its decomposition into irreducible components.

The details will appear elsewhere.

**2.** The triple $(\mathfrak{k}, j, \rho)$ consisting of an exponential Lie algebra $\mathfrak{k}$, a linear operator $j$ and an alternating bilinear form $\rho$ on $\mathfrak{k}$ is called an exponential Kähler algebra if it has the following properties:

a) $j^2=-1$, b) $[jX, jY]=j[jX, Y]+j[X, jY]+[X, Y]$,

c) $\rho(jX, jY)=\rho(X, Y)$, d) $\rho(jX, X)>0$ for $X\neq 0$,

e) $\rho([X, Y], Z)+\rho([Y, Z], X)+\rho([Z, X], Y)=0$.

If, in addition to these properties, there is an linear form $\omega$ on $\mathfrak{k}$ such that $\rho(X, Y)=\omega([X, Y])$ for any $X, Y \in \mathfrak{k}$, the triple $(\mathfrak{k}, j, \omega)$ is called an exponential $j$-algebra. By abuse of language we often call the exponential Lie algebra $\mathfrak{k}$ an exponential Kähler algebra or an exponential $j$-algebra.

We generalize the structure theorem of a normal $j$-algebra [4] (resp. a normal Kähler algebra [3]) to an exponential $j$-algebra (resp. an exponential Kähler algebra).

**Theorem 1.** *Let* $(\mathfrak{k}, j, \omega)$ *be an exponential j-algebra. We define an inner product S on* $\mathfrak{k}$ *by* $S(X, Y)=\omega([jX, Y])$ *for* $X, Y \in \mathfrak{k}$. *Let* $\mathfrak{a}$ *be the orthogonal complement of* $\eta=[\mathfrak{k}, \mathfrak{k}]$ *with respect to the form S.* $\mathfrak{a}$ *is a commutative subalgebra of* $\mathfrak{k}$, $\mathfrak{k}=\mathfrak{a}+\eta$, *and the adjoint representation of* $\mathfrak{a}$ *on* $\eta$ *is complex diagonarizable. For* $\alpha \in \mathfrak{a}^*$, *we set* $\eta^\alpha=\{X \in \eta ; [A, X]=\alpha(A)X$ *for all* $A \in \mathfrak{a}\}$ *and let* $\{\eta^{\alpha_i}\}$, $1\leq i\leq r$ *be those root spaces* $\eta^\alpha$ *for which* $j(\eta^\alpha)\subset \mathfrak{a}$. *Then* $\dim \eta^{\alpha_i}=1$ *and* $r=\dim \mathfrak{a}$ *(r is called* *the rank of* $\mathfrak{k}$). *If we order* $\alpha_1, \cdots, \alpha_r$ *in an appropriate way, then all the other real roots are of the form*

$$\frac{1}{2}(\alpha_m+\alpha_k), \quad \frac{1}{2}(\alpha_m-\alpha_k), \quad 1\le k<m\le r,$$

$$\frac{1}{2}\alpha_k, \qquad\qquad 1\le k\le r$$

(*not all possibilities need occur*), *and* $\eta$ *can be decomposed as follows:*

$$\eta=\sum_{m>k}\eta^{1/2(\alpha_m-\alpha_k)}+\mathfrak{k}_{1/2}+\sum_{m\geqslant k}\eta^{1/2(\alpha_m+\alpha_k)},$$

*where* $\mathfrak{k}_{1/2}=\sum_k \tilde{\eta}^{1/2\alpha_k}$ *and* $\tilde{\eta}^{1/2\alpha_k}$ *is an* ad $\mathfrak{a}$-*invariant subspace, the complexification of which is the sum of root spaces of* ad $\mathfrak{a}$ *with roots* $A\mapsto\frac{1}{2}\alpha_k(A)(1+i\beta_{k,p})$ $(A\in\mathfrak{a})$ *with* $\beta_{k,p}\in \boldsymbol{R}$. *Let* $\mathfrak{k}_0=\mathfrak{a}+\sum_{m>k}\eta^{1/2(\alpha_m-\alpha_k)}$, $\mathfrak{k}_1=\sum_{m\geqslant k}\eta^{1/2(\alpha_m+\alpha_k)}$, *then* $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{k}_{1/2}+\mathfrak{k}_1$, $[\mathfrak{k}_i,\mathfrak{k}_k]\subset\mathfrak{k}_{i+k}$, $j(\eta^{1/2(\alpha_m-\alpha_k)})=\eta^{1/2(\alpha_m+\alpha_k)}$, $r\ge m>k\ge 1$, $j(\tilde{\eta}^{1/2\alpha_k})=\tilde{\eta}^{1/2\alpha_k}$ $r\ge k\ge 1$. *Let* $U_i$ *be the nonzero element of* $\eta^{\alpha_i}$ *such that* $[jU_i,U_i]=U_i$ *and let* $s=\sum_{i=1}^r U_i$. *Then* $\alpha_k(jU_i)=\delta_{k,i}$, ad $js|\mathfrak{k}_0=0$, ad $js|\mathfrak{k}_1=Id$, ad $js|\mathfrak{k}_{1/2}$ *is semisimple and its eigenvalues have the real part* $\frac{1}{2}$. *We have* $jX=[s,X]$ *for* $X\in\mathfrak{k}_0$.

**Theorem 2.** *Let* $\mathfrak{k}$ *be an exponential Kähler algebra, then* $\mathfrak{k}$ *can be decomposed into a semi-direct sum* $\mathfrak{k}=\mathcal{J}+\mathcal{H}$, *where* $\mathcal{J}$ *is a commutative* $j$-*invariant ideal, and* $\mathcal{H}$ *is an exponential* $j$-*algebra.*

**3.** Now we return to the problems stated in §1. For a real vector space $V$, we denote its dual by $V^*$. For an exponential Lie algebra $\mathfrak{k}$ and $l\in\mathfrak{k}^*$, we denote by $P^+(l,\mathfrak{k})$ the set of positive polarizations of $\mathfrak{k}$ at $l$. Let $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$, $\mathfrak{e}=(\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{g}$ and let $\mathfrak{b}=\mathfrak{d}\cap\ker f$. $\mathfrak{d}$ and $\mathfrak{b}$ are ideals of $\mathfrak{e}$. Let $\tilde{\mathfrak{e}}=\mathfrak{e}/\mathfrak{b}$, $\mathfrak{z}=\mathfrak{d}/\mathfrak{b}$, $\pi:\mathfrak{e}\to\tilde{\mathfrak{e}}$ the projection, $f_0=f|\mathfrak{e}\in\mathfrak{e}^*$, $\tilde{\mathfrak{h}}=\pi(\mathfrak{h})$ and let $\tilde{f}\in(\tilde{\mathfrak{e}})^*$ such that $\tilde{f}\circ\pi=f_0$.

**Theorem 3.** $\tilde{\mathfrak{e}}$ *can be decomposed into a semi-direct sum*

$$\tilde{\mathfrak{e}}=\mathfrak{n}+\mathfrak{m},\quad \mathfrak{m}: \textit{subalgebra},\quad \mathfrak{n}: \textit{ideal},$$

*and this decomposition satisfies the following conditions.*

*Let* $\mathfrak{h}_1=\tilde{\mathfrak{h}}\cap\mathfrak{n}_C$, $\mathfrak{h}_2=\tilde{\mathfrak{h}}\cap\mathfrak{m}_C$, $\tilde{f}_1=\tilde{f}|\mathfrak{n}\in\mathfrak{n}^*$ *and let* $\tilde{f}_2=\tilde{f}|\mathfrak{m}\in\mathfrak{m}^*$.

a) $\mathfrak{n}$ *is a Heisenberg algebra with center* $\mathfrak{z}$ *and* $\mathfrak{h}_1\in P^+(\tilde{f}_1,\mathfrak{n})$.

b) $\mathfrak{h}_2\in P^+(\tilde{f}_2,\mathfrak{m})$ and $\mathfrak{h}_2+\bar{\mathfrak{h}}_2=\mathfrak{m}_C$, $\mathfrak{h}_2\cap\mathfrak{m}=\{0\}$. *We define the linear operator* $j$ *on* $\mathfrak{m}$ *by* $j(X)=-iX$ *if* $X\in\mathfrak{h}_2$, $j(X)=iX$ *if* $X\in\bar{\mathfrak{h}}_2$. *Then* $(\mathfrak{m},j,-\tilde{f}_2)$ *is an exponential* $j$-*algebra.*

**4.** We use the notations of Theorem 1 applied to $\mathfrak{m}$. Let $L_i=\sum_{j>i}\eta^{1/2(\alpha_j-\alpha_i)}$, $L_i'=\sum_{i>j}\eta^{1/2(\alpha_i-\alpha_j)}$, $p_i=\dim L_i'$, $q_i=\dim L_i$, $r_i=\dim\tilde{\eta}^{1/2\alpha_i}$ and let $f_i=\tilde{f}_2(U_i)$, $1\le i\le r$. Let $W=\ker\tilde{f}_1\subset\mathfrak{n}$. Then $W$ is invariant under $\mathrm{ad}_{\mathfrak{n}}\,\mathfrak{m}$, $\mathrm{ad}_{W_C}\,\mathfrak{a}$ is diagonalizable and $W_C$ can be decomposed into root spaces $(W_C)^\beta$ with roots of the form $\beta(A)=\pm\frac{1}{2}\alpha_k(A)(1+i\beta'_{k,l})$ $(A\in\mathfrak{a})$, $\beta'_{k,l}\in\boldsymbol{R}$ or $\beta=0$ (not all possibilities need occur). We put $\tilde{W}_C^{\alpha_k/2}=\sum_{\beta=1/2(1+i\beta'_{k,l})\alpha_k}(W_C)^\beta$ and put $\tilde{W}^{\alpha_k/2}=\tilde{W}_C^{\alpha_k/2}\cap W$ $(1\leq k\leq r)$. Let $t_k=\dim\tilde{W}^{\alpha_k/2}$ $(1\leq k\leq r)$.

By modifying the result and the method of Rossi-Vergne [5], we obtain the following theorem.

**Theorem 4.** *$\mathcal{H}(f,\mathfrak{h})\neq\{0\}$ if and only if*

$$-2f_i-\left(p_i+1+\frac{1}{2}(q_i+r_i+t_i)\right)>0,\qquad 1\leq i\leq r.$$

*The last inequality is identical with the result of Rossi-Vergne* [5], *except the appearance of the term $t_i$.*

**5.** $G$ acts on $\mathfrak{g}^*$ by the coadjoint representation and thus we have the orbit space $\mathfrak{g}^*/G$. We denote by $O(f)$ the orbit through $f$. For each orbit $\sigma\in\mathfrak{g}^*/G$, we denote by $\hat{\rho}(\sigma)$ the equivalence class of irreducible unitary representations of $G$ associated to $\sigma$ in the sense of Kirillov-Bernat. For each subspace $\mathfrak{p}$ of $\mathfrak{g}$, we set $\mathfrak{p}^\perp=\{g\in\mathfrak{g}^*\,;\,g|\mathfrak{p}=0\}$. Let $D=\exp\mathfrak{d}$. We say that $\mathfrak{h}$ satisfies the Pukanszky condition if $D.f=f+\mathfrak{e}^\perp$.

**Theorem 5.** *Suppose $\mathcal{H}(f,\mathfrak{h})\neq\{0\}$. Then $\rho(f,\mathfrak{h})$ is irreducible if and only if $\mathfrak{h}$ satisfies the Pukanszky condition. In this case, $\rho(f,\mathfrak{h})\simeq\hat{\rho}(O(f))$. In particular, $\rho(f,\mathfrak{h})$ is independent of $\mathfrak{h}$.*

**6.** We denote by $U(f,\mathfrak{h})$ the set of orbits $\sigma\in\mathfrak{g}^*/G$ such that $\sigma\cap(f+\mathfrak{e}^\perp)$ is non-empty open set in $f+\mathfrak{e}^\perp$. For $\sigma\in\mathfrak{g}^*/G$, we denote by $c(\sigma,f,\mathfrak{h})$ the number of connected components of $\sigma\cap(f+\mathfrak{e}^\perp)$. Then we have the following theorem, which generalizes the result of M. Vergne [6] for real polarizations.

**Theorem 6.** *If $\mathcal{H}(f,\mathfrak{h})\neq\{0\}$, then*

a) *$U(f,\mathfrak{h})$ is a finite set.*

b) *For $\sigma\in U(f,\mathfrak{h})$, $c(\sigma,f,\mathfrak{h})<+\infty$.*

c) $\rho(f,\mathfrak{h})\simeq\sum_{\sigma\in U(f,\mathfrak{h})}c(\sigma,f,\mathfrak{h})\hat{\rho}(\sigma)$.

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