135. *Tensor Products of Positive Definite Quadratic Forms*

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Let L, M, N be positive definite quadratic lattices over $\boldsymbol{Z}$. Our aim is to give some affirmative answers for the following two problems:

i) If M, N are indecomposable, then is $M \otimes N$ indecomposable?

ii) If $L \otimes M$ is isometric to $L \otimes N$, then is M isometric to N?

Definitions and notations. By a positive definite quadradic lattice we mean a lattice L of a positive definite quadratic space V over the rational number field $\boldsymbol{Q}$ ($\operatorname{rank} L = \dim V$).

Let L be a positive definite quadradic lattice; then $m(L)$ denotes $\min Q(x)$, where Q is the quadratic form of L and x runs over non-zero elements of L, and moreover we call an element x of L a minimal vector of L if $Q(x) = m(L)$. $\mathfrak{m}(L)$ denotes the set of all minimal vectors of L, and $\tilde{L}$ is by definition the submodule of L spanned by all minimal vectors of L.

Let L, M be positive definite quadratic lattices with bilinear forms B_L, B_M respectively. Then the tensor product $L \otimes M$ over $\boldsymbol{Z}$ is a positive definite quadratic lattice with bilinear form B such that $B(x_1 \otimes y_1, x_2 \otimes y_2) = B_L(x_1, x_2) B_M(y_1, y_2)$ for any $x_i \in L$, $y_i \in M$.

Through this note $Q(x)$, $B(x, y)$ denote quadratic forms and corresponding bilinear forms ($2B(x, y) = Q(x+y) - Q(x) - Q(y)$), and notations and terminologies will be those of O'Meara [2].

§ 1. Positive definite quadratic lattices of E-type and their properties.

Definition. Let L be a positive definite quadratic lattice. We say that L is of E-type if every minimal vector of $L \otimes M$ is of the form $x \otimes y$ ($x \in L, y \in M$) for any positive definite quadratic lattice M.

Theorem. (i) *If L_1, L_2 are of E-type*, then $L_1 \perp L_2$, $L_1 \otimes L_2$ are of E-type.*

(ii) *If L is of E-type and if L_1 is a submodule of L with $m(L_1) = m(L)$, then L_1 is of E-type.*

(iii) *If L is a positive definite quadratic lattice such that either $m(L) \leq 6$ and the scale sL of $L \subseteq \boldsymbol{Z}$, or rank $L \leq 42$, then L is of E-type.*

This is proved in [1].

§ 2. **Theorem.** *Let L be an indecomposable positive definite*

*) When we say that L is of E-type, L is assumed to be a poitive definite quadratic lattice.

quadratic lattice of E-type with $[L;\tilde{L}]<\infty$. *Then for any indecomposable positive definite quadratic lattice* M, $L\otimes M$ *is indecomposable.*

Lemma. *Let* L, M, N *be positive definite quadratic lattices and assume that* L *is indecomposable. Suppose that* σ *is an isometry from* $L\otimes M$ *on* $L\otimes N$ *satisfying that*

(*) *there are sublattices* M' *of* M *and* N' *of* N *with* $[M:M']<\infty$, $[N:N']<\infty$ *such that* $M'=M_1'\perp\cdots\perp M_n'$, $N'=N_1'\perp\cdots\perp N_n'$ *and* $\sigma=\sigma_i'\otimes\mu_i'$ *on* $L\otimes M_i'$ *where* $\sigma_i'\in 0(L)$ *and* μ_i' *is an isometry from* M_i' *on* N_i'.

Then there are decompositions $M=M_1\perp\cdots\perp M_m, N=N_1\perp\cdots\perp N_m$ *such that* $\sigma=\sigma_i\otimes\mu_i$ *on* $L\otimes M_i$ *where* $\sigma_i\in 0(L)$ *and* μ_i *is an isometry from* M_i *on* N_i.

Lemma. *Let* L, M, N *be positive definite quadratic lattices, and assume that* $[L:\tilde{L}], [M:\tilde{M}], [N:\tilde{N}]<\infty$ *and* $\tilde{L}, \tilde{M}, \tilde{N}$ *are indecomposable, and that either* M, N *are of E-type or* L *is of E-type. If an isometry* σ *from* $L\otimes M$ *on* $L\otimes N$ *satisfies* $\sigma(L\otimes u)=L\otimes v$ *for some* $0\neq u\in M$, $v\in N$ *or* $\sigma(l\otimes M)=l'\otimes N$ *for some* $0\neq l$, $l'\in L$, *then* $\sigma=\alpha\otimes\mu$ *where* $\alpha\in 0(L)$, μ *is an isometry from* M *on* N.

Definition. For positive definite quadratic lattices M, the definition of "generic" is defined inductively as follows: If $\operatorname{rank} M=1$, then M is generic. When $\operatorname{rank} M\geq 2$, M is generic if and only if $\mathfrak{m}(M)=\{\pm u\}$ and $u^{\perp}$ is generic.

Theorem. *Let* L *be an indecomposable positive definite quadratic lattice of E-type with* $[L:\tilde{L}]<\infty$. *Let* M, N *be positive definite quadratic lattices and assume that* M *is generic. For any isometry* σ *from* $L\otimes M$ *on* $L\otimes N$, *there are decompositions* $M=\perp_{i=1}^{n}M_i$, $N=\perp_{i=1}^{n}N_i$ *such that* $\sigma=\sigma_i\otimes\mu_i$ *on* $L\otimes M_i$ *where* $\sigma_i\in 0(L)$ *and* μ_i *is an isometry from* M_i *on* N_i.

Theorem. *Let* L, M, N *be positive definite quadratic lattices with* $[L:\tilde{L}], [M:\tilde{M}], [N:\tilde{N}]<\infty$, *and let* $\tilde{L}, M, N$ *be indecomposable. Assume that* M, N *are of E-type or* L *is of E-type, and moreover*

$$\{|B(x,y)|/m(L); x,y\in\mathfrak{m}(L)\}\cap\{|B(x,y)|/m(M); x,y\in\mathfrak{m}(M)\}\subset\{0,1\},$$

$$\{|B(x,y)|/m(L); x,y\in\mathfrak{m}(L)\}\cap\{|B(x,y)|/m(N); x,y\in\mathfrak{m}(N)\}\subset\{0,1\}.$$

Then an isometry σ *from* $L\otimes M$ *on* $L\otimes N$ *is of the form* $\alpha\otimes\mu$ *where* $\alpha\in 0(L)$, μ *is an isometry* M *on* N.

Theorem. *Let* L *be a positive definite quadratic lattice with the scale* $sL=Z$ *and* $m(L)=2$. *Assume that* $\tilde{L}$ *is indecomposable and* $[L:\tilde{L}]<\infty$. *Let* M, N *be positive definite quadratic lattices with* $[M;\tilde{M}], [N:\tilde{N}]<\infty$.

Suppose that σ *is an isometry from* $L\otimes M$ *on* $L\otimes N$. *Then for any* $u\in\mathfrak{m}(M)$, *we have* i) $\sigma(L\otimes u)=L\otimes v$ *for some* $v\in\mathfrak{m}(N)$, *or* ii) $\sigma(L\otimes u)\subset w\otimes N$ *for some* $w\in\mathfrak{m}(L)$. *If, moreover, the case* i) *does not happen for any* $u\in\mathfrak{m}(M)$, *then* M, N *are isometric to* $L\otimes K_1$, $L\otimes K_2$ *respectively where* K_1, K_2 *are positive definite quadratic lattices.*

Remark. Theorem is not true without the assumptions $sL=\mathbf{Z}$, $m(L)=2$.

Corollary 1. *Let L be a quadratic lattice in Theorem, and let M, N be positive definite quadratic lattices with $\operatorname{rank} M \leq \operatorname{rank} L$. If σ is an isometry from $L\otimes M$ on $L\otimes N$, then M is isometric to N and moreover there are decompositions $M=M_1\perp\cdots\perp M_n$, $N=N_1\perp\cdots\perp N_n$ such that $\sigma=\sigma_i\otimes\mu_i$ on $L\otimes M_i$ where $\sigma_i\in O(L)$ and μ_i is an isometry from M_i on N_i unless $M\cong N\cong L^a$ (scaling of L).*

Corollary 2. *Let L be an indecomposable positive definite quadratic lattice with $sL=\mathbf{Z}$, $m(L)=2$ and $[L:\tilde{L}]<\infty$. Let M,N be binary positive definite quadratic lattices. If $L\otimes M$ is isometric to $L\otimes N$, then M is isometric to N.*

Corollary 3. *Let L be a positive definite quadratic lattice in Theorem. Let M,N be positive definite quadratic lattices. Assume that $\tilde{M}$ is indecomposable with $[M:\tilde{M}]<\infty$, and M is not of the form $L\otimes K$. If σ is an isometry from $L\otimes M$ on $L\otimes N$, then $\sigma=\alpha\otimes\mu$ where $\alpha\in O(L)$ and μ is an isometry from M on N.*

The detailed proof will be published elsewhere.

References

[1] Y. Kitaoka: Scalar extension of quadratic lattice. II (to appear).
[2] O. T. O'Meara: Introduction to Quadratic Forms. Springer-Verlag (1963).